Canad. Math. Bull. Vol. **56** (4), 2013 pp. 785–794
http://dx.doi.org/10.4153/CMB-2012-025-0

Small Prime Solutions to Cubic Diophantine Equations

Zhixin Liu

Abstract. Let $a_1, \dots, a_9$ be nonzero integers and n any integer. Suppose that $a_1+\cdots+a_9 \equiv n \pmod 2$ and $(a_i, a_j) = 1$ for $1 \le i < j \le 9$. In this paper we prove the following:

(i) If a_j are not all of the same sign, then the cubic equation $a_1 p_1^3 + \cdots + a_9 p_9^3 = n$ has prime solutions satisfying $p_j \ll |n|^{1/3} + \max\{|a_j|\}^{14+\varepsilon}$.

(ii) If all a_j are positive and $n \gg \max\{|a_j|\}^{43+\varepsilon}$, then $a_1 p_1^3 + \cdots + a_9 p_9^3 = n$ is solvable in primes p_j.

These results are an extension of the linear and quadratic relative problems.

1 Introduction

For any integer n, we consider cubic equations in the form

$$a_1 p_1^3 + \cdots + a_9 p_9^3 = n, \tag{1.1}$$

where p_j are prime variables and the coefficients a_j are nonzero integers. A necessary condition for the solvability of (1.1) is

$$a_1 + \cdots + a_9 \equiv n \pmod 2. \tag{1.2}$$

We also suppose

$$(a_i, a_j) = 1, \quad 1 \le i < j \le 9, \tag{1.3}$$

and write $A = \max\{2, |a_1|, \dots, |a_9|\}$. The main results in this paper are the following two theorems.

Theorem 1.1 *Suppose (1.2) and (1.3) hold. If $a_1, \dots, a_9$ are not all of the same sign, then (1.1) has solutions in primes p_j satisfying*

$$p_j \ll |n|^{1/3} + A^{14+\varepsilon},$$

where the implied constant depends only on ε.

Theorem 1.2 *Suppose (1.2) and (1.3) hold. If $a_1, \dots, a_9$ are all positive, then (1.1) is solvable whenever*

$$n \gg A^{43+\varepsilon},$$

where the implied constant depends only on ε.

Received by the editors December 18, 2011; revised May 4, 2012.
Published electronically August 27, 2012.
AMS subject classification: **11P32**, 11P05, 11P55.
Keywords: small prime, Waring–Goldbach problem, circle method.

Theorems 1.1 and 1.2 are proved by the circle method. Instead of the iterative argument, we use a new idea introduced by J. Y. Liu [16] (see Section 4 below) to enlarge the major arcs. In this process, we get the larger major arcs in the circle method.

Theorem 1.2 with $a_1 = \cdots = a_9 = 1$ is a classical result of Hua [9] from 1938. Our investigation on (1.1) is also motivated by the following works.

In his well-known work [1], Baker first raised and investigated the problem of small prime solutions of the equation

$$a_1 p_1 + a_2 p_2 + a_3 p_3 = n,$$

satisfying

$$|a_j| p_j \ll |n| + A^C, \tag{1.4}$$

where a_1, a_2, a_3, n are nonzero integers satisfying some necessary conditions, and $A = \max\{2, |a_1|, |a_2|, |a_3|\}$. This problem was later settled qualitatively by M. C. Liu and Tsang [14]. Choi [2] proved that $C = 4190$ in (1.4), and M. C. Liu and Wang [15] improved this to $C = 45$, and then Li [12] to $C = 38$. Under the Generalized Riemann Hypothesis, Choi, M. C. Liu, and Tsang [7] reduced the constant to $C = 5+\varepsilon$. J. Y. Liu and Tsang [18] showed that when the necessary conditions in this problem are replaced by some more restrictive conditions, one can take $C = 17/2$. With the same restrictive conditions as in [18], Choi and Kumchev [3] further reduced this to $C = 20/3$.

M. C. Liu and Tsang [13] first studied the quadratic equation

$$a_1 p_1^2 + \cdots + a_5 p_5^2 = n,$$

satisfying

$$p_j \ll |n|^{1/2} + A^C, \tag{1.5}$$

where $a_1, \ldots, a_5, n$ are nonzero integers satisfying some necessary conditions, and $A = \max\{2, |a_1|, \ldots, |a_5|\}$. The first numerical result for C in (1.5) was $C = 20 + \varepsilon$, obtained by Choi and J. Y. Liu [6]. The number 20 was subsequently reduced to $25/2$ by Choi and J. Y. Liu [5] and then to 8 by Choi and Kumchev [4]. The best result is due to Harman and Kumchev [8] who showed that $C = 7$.

Theorems 1.1 and 1.2 improve the results in [11] with the bounds $20+\varepsilon$ and $61+\varepsilon$ in place of $14 + \varepsilon$ and $43 + \varepsilon$, respectively.

In general, if we only assume $(a_1, a_2, \ldots, a_9) = 1$, then the proof of the solvability result of (1.1) is complicated and relies on the explicit zero-free regions of Dirichlet L-functions and Deuring–Heilbronn phenomenon. This usually gives unsatisfactory results. In this paper, we assume the somewhat stricter condition $(a_i, a_j) = 1$ for $1 \le i < j \le 9$, and the proof will be much simplified and won't involve the explicit zero-free region and Deuring–Heilbronn phenomenon. In this process, some effective techniques (see Section 4 below, or [17]) for treating the major arcs can be used.

https://doi.org/10.4153/CMB-2012-025-0 Published online by Cambridge University Press

Notation As usual, $\varphi(n)$ stands for the function of Euler, and $d(n)$ is the divisor function. We use $\chi \bmod q$ and $\chi^0 \bmod q$ to denote a Dirichlet character and the principal character modulo q, respectively. $r \sim R$ means $R < r \leq 2R$. The letter c denotes an absolute positive constant that may vary at different places. The letter ε denotes a positive constant that is arbitrarily small. We also write $(a, \dots, b) = \gcd(a, \dots, b)$. For this paper, we set $N_j = (N/a_j)^{1/3}$.

2 Outline of the Method

Denote by $r(n)$ the weighted number of solutions of (1.1), *i.e.*,

$$r(n) = \sum_{\substack{n=a_1p_1^3+\cdots+a_9p_9^3 \\ M<|a_j|p_j^3\leq N}} (\log p_1)\cdots(\log p_9),$$

where $M = N/200$. We will investigate $r(n)$ by the circle method. To this end, we set

$$P = (N/A)^{3/13-\varepsilon}, \quad Q = N^{1-2\varepsilon}P^{-1}, \quad \text{and} \quad L = \log N. \tag{2.1}$$

By Dirichlet's lemma on rational approximation, each $\alpha \in [1/Q, 1 + 1/Q]$ may be written in the form

$$\alpha = a/q + \lambda, \quad |\lambda| \leq 1/(qQ) \tag{2.2}$$

for some integers a, q with $1 \leq a \leq q \leq Q$ and $(a, q) = 1$. We denote by $\mathfrak{M}(a, q)$ the set of α satisfying (2.2), and define the major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$ as follows:

$$\mathfrak{M} = \mathfrak{M}(P) = \bigcup_{q\leq P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}(a, q), \quad \mathfrak{m} = \Big[\frac{1}{Q}, 1 + \frac{1}{Q}\Big] \setminus \mathfrak{M}. \tag{2.3}$$

It follows from $2P \leq Q$ that the major arcs $\mathfrak{M}(a, q)$ are mutually disjoint. Let

$$S_j(\alpha) = \sum_{M<|a_j|p^3\leq N} (\log p)e(a_jp^3\alpha).$$

Then we have

$$r(n) = \int_0^1 S_1(\alpha)\cdots S_9(\alpha)e(-n\alpha)\,d\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}}.$$

The integral on the major arcs $\mathfrak{M}$ causes the main difficulty, which is solved by the following.

Theorem 2.1 *Assume* (1.3). *Let $\mathfrak{M}$ be as in* (2.3) *with P, Q determined by* (2.1). *Then we have*

$$\int_{\mathfrak{M}} S_1(\alpha)\cdots S_9(\alpha)e(-n\alpha)\,d\alpha = \frac{1}{3^9}\mathfrak{S}(n, P)\mathfrak{J}(n) + O\Big(\frac{N^2}{|a_1\cdots a_9|^{1/3}L}\Big), \tag{2.4}$$

where $\mathfrak{S}(n, P)$ and $\mathfrak{J}(n)$ are defined in (2.6) *and* (2.7) *respectively.*

https://doi.org/10.4153/CMB-2012-025-0 Published online by Cambridge University Press

The starting point of our proof of Theorem 2.1 is Lemma 2.2, which deals with major arcs of classical size. Let

$$P_0 = N^{\varepsilon}, \quad Q_0 = N^{1-2\varepsilon}. \tag{2.5}$$

Define the major arcs $\mathfrak{M}_0 = \mathfrak{M}(P_0)$ as in (2.3). The following lemma is now standard by the iterative method introduced by J. Y. Liu [17]. The proof of Lemma 2.2 for quadratics can be found in [5]. The size of major arcs of Theorem 3 in [5] is larger than that of Lemma 2.2 below, so we can enlarge major arcs of Lemma 2.2 by the method in [5], but our choice of the size of major arcs in Lemma 2.2 is strong enough. Thus, the proof of Lemma 2.2 is omitted.

Lemma 2.2 *Let $B > 0$ be sufficiently large, then*

$$\int_{\mathfrak{M}_0} S_1(\alpha)\cdots S_9(\alpha)e(-n\alpha)\,d\alpha = \frac{1}{3^9}\mathfrak{S}(n,P)\mathfrak{J}(n) + O\Big(\frac{N^2}{|a_1\cdots a_9|^{1/3}L}\Big)$$

where $\mathfrak{S}(n,P)$ and $\mathfrak{J}(n)$ are the same as those in Theorem 2.1.

To derive Theorem 2.1, we need to bound $\mathfrak{S}(n,P)$ and $\mathfrak{J}(n)$ from below. For $\chi \bmod q$, we define

$$C(\chi,a) = \sum_{h=1}^{q} \overline{\chi}(h)e\Big(\frac{ah^3}{q}\Big), \quad C(q,a) = C(\chi^0,a).$$

If $\chi_1,\dots,\chi_9$ are characters $\bmod q$, then we write

$$B(n,q,\chi_1,\dots,\chi_9) = \sum_{h=1\,(h,q)=1}^{q} e\Big(-\frac{hn}{q}\Big)C(\chi_1,a_1h)\cdots C(\chi_9,a_9h),$$

$$B(n,q) = B(n,q,\chi^0,\dots,\chi^0), \quad A(n,q) = \frac{B(n,q)}{\varphi^9(q)},$$

and

$$\mathfrak{S}(n,P) = \sum_{q\le P} A(n,q). \tag{2.6}$$

Lemma 2.3 *Assuming (1.2), we have $\mathfrak{S}(n,P) \gg (\log\log A)^{-c}$ for some constant $c > 0$.*

Lemma 2.4 *Suppose (1.3) and*

(i) *$a_1,\dots,a_9$ are not all of the same sign and $N \ge 27|n|$; or*
(ii) *$a_1,\dots,a_9$ are positive and $n = N$.*

Then we have

$$\mathfrak{J}(n) := \sum_{\substack{a_1m_1+\cdots+a_9m_9=n\\ M<|a_j|m_j\le N}} (m_1\cdots m_9)^{-2/3} \asymp \frac{N^2}{|a_1\cdots a_9|^{1/3}}. \tag{2.7}$$

We remark that Lemma 2.3 and Lemma 2.4 can be treated mostly as the same as those in [6]. Thus the proofs are omitted, and therefore we may concentrate on (2.4) in the following sections.

https://doi.org/10.4153/CMB-2012-025-0 Published online by Cambridge University Press

3 Some Lemmas

We derive estimates for the generating functions appearing in the proof from estimates for the exponential sum

$$S(\alpha)=\sum_{X<p\le 2X}(\log p)e(\alpha p^3), \tag{3.1}$$

which are given in terms of the rational approximation

$$\alpha=\frac{a}{q}+\lambda,\quad \text{with}\quad 1\le a\le q,\quad (a,q)=1. \tag{3.2}$$

We start by quoting the results of Ren [19] and Kumchev [10].

Lemma 3.1 *Let α satisfy (3.2). Then*

$$S(\alpha)\ll\Big(X^{1/2}\sqrt{q(1+|\lambda|X^3)}+X^{4/5}+\frac{X}{\sqrt{q(1+|\lambda|X^3)}}\Big)q^{\varepsilon}\log^c X,$$

where $\varepsilon>0$ is a constant arbitrarily small, and $c>0$ an absolute constant.

Lemma 3.2 *Suppose that $\alpha\in\mathbb{R}$ and that exist $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfying*

$$1\le q\le Q,\quad (a,q)=1,\quad |q\alpha-a|<Q^{-1}$$

with

$$Q=X^{12/7}.$$

Then, for any fixed $\varepsilon>0$,

$$S(\alpha)\ll\Big(X^{13/14+\varepsilon}+\frac{X^{1+\varepsilon}}{\sqrt{q(1+|\lambda|X^3)}}\Big),$$

where the implied constant depends at most on k and ε.

The next two lemmas generalize Lemma 3.1 and Lemma 3.2 to $S(b\alpha)$, with b a nonzero integer.

Lemma 3.3 *Let b be a nonzero integer and let $S(\alpha)$ be defined by (3.1). Suppose that there exist $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfying*

$$1\le q\le P,\quad (a,q)=1,\quad |q\alpha-a|<P/(|b|X^3), \tag{3.3}$$

with $P<X/2$. Then, for any fixed $\varepsilon>0$, we have

$$S(b\alpha)\ll\big(X^{1/2}\Phi(\alpha)^{1/2}+X^{4/5}+X\Phi(\alpha)^{-1/2}\big)q^{\varepsilon}\log^c X,$$

where $\Phi(\alpha)=q_1(1+|b|X^3|\alpha-a/q|)$ and $q_1=q/(b,q)$.

Proof By Dirichlet's theorem on diophantine approximation, there exist integers a_1 and q_1 satisfying

$$1 \le q_1 \le X, \quad (a_1, q_1) = 1, \quad |q_1 b\alpha - a_1| < X^{-1}. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$|q_1 ba - qa_1| \le q_1 |b|\, |q\alpha - a| + q|q_1 b\alpha - a_1| \le 2PX^{-1} < 1,$$

and hence

$$\frac{a_1}{q_1} = \frac{ab}{q}, \quad \text{and} \quad q_1 = \frac{q}{(q, b)}.$$

Thus

$$\Phi(\alpha) = q_1 + X^3 |q_1 b\alpha - a_1|,$$

and the lemma follows from Lemma 3.1 with $\alpha = b\alpha$, $q = q_1$, and $a = a_1$.

Lemma 3.4 *Let b be a nonzero integer and let $S(\alpha)$ be defined by (3.1). Suppose that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying*

$$1 \le q \le |b| X^3 P^{-1}, \quad (a, q) = 1, \quad |q\alpha - a| < P/(|b|X^3), \tag{3.5}$$

with P subject to

$$2|b|X^{1/7} < P \le X. \tag{3.6}$$

Then, for any fixed $\varepsilon > 0$, we have

$$S(b\alpha) \ll X^{13/14+\varepsilon} + X^{1+\varepsilon} \Phi(\alpha)^{-1/2}, \tag{3.7}$$

where $\Phi(\alpha) = q_1(1 + |b|X^3 |\alpha - a/q|)$ and $q_1 = q/(b, q)$.

Proof By Dirichlet's theorem, there exist integers a_1 and q_1 such that

$$1 \le q_1 \le X^{12/7}, \quad (a_1, q_1) = 1, \quad |q_1 b\alpha - a_1| < X^{-12/7}.$$

Hence, by Lemma 3.2 with $\alpha = b\alpha$, $q = q_1$, and $a = a_1$,

$$S(b\alpha) \ll X^{13/14+\varepsilon} + \frac{X^{1+\varepsilon}}{\sqrt{q_1 + X^3 |q_1 b\alpha - a_1|}}. \tag{3.8}$$

If $q_1 > X^{1/7}$ or $|q_1 b\alpha - a_1| > X^{-20/7}$, the first term on the right side of (3.8) dominates the second and (3.7) follows. Otherwise, recalling (3.5) and (3.6), we get

$$|q_1 ba - aq_1| \le q_1 |b|\, |q\alpha - a| + q|q_1 b\alpha - a_1| \le PX^{-20/7} + |b| X^{1/7} P^{-1} < 1.$$

Thus

$$\frac{a_1}{q_1} = \frac{ab}{q} \quad \text{and} \quad q_1 = \frac{q}{(q, b)},$$

and (3.8) turns into (3.7).

https://doi.org/10.4153/CMB-2012-025-0 Published online by Cambridge University Press

4 Enlarge the Major Arcs and the Proof of Theorem 2.1

To establish (2.4), we apply Lemma 2.2, which states that the integral on $\mathfrak{M}_0$ already gives the desired asymptotic formula, and therefore it remains to show that the integral on $\mathfrak{M} \setminus \mathfrak{M}_0$ is small. To achieve this, we are going to apply Lemma 3.3 to each $S_i(\alpha)$ on $\mathfrak{M} \setminus \mathfrak{M}_0$, but before doing so, we must understand the structure of $\mathfrak{M} \setminus \mathfrak{M}_0$, which is best seen through dyadic divisions.

Denote by $E(K)$ the set of $\alpha \in [0,1]$ satisfying

$$\alpha = \frac{a}{q} + \lambda, \quad (a,q) = 1, \quad 1 \le a \le q \le K, \quad |\lambda| \le \frac{K}{qN}.$$

Let P_0 and P be as in (2.5) and (2.1) respectively, and write $P_j = 2^j P_0$ for $j = 1, 2, \dots$ so that

$$P_0 < P_1 < \cdots < P_{h-1} < P \le P_h$$

for some $h \ll \log X$. We observe that every $\alpha \in \mathfrak{M}$ lies in $E(P_h)$, and

$$\mathfrak{M} \setminus \mathfrak{M}_0 \subset \bigcup_{j=1}^{h} \{E(P_j) \setminus E(P_{j-1})\}. \tag{4.1}$$

By construction, every $\alpha \in E(P_j) \setminus E(P_{j-1})$ has a Diophantine approximation $\alpha = a/q + \lambda$ with

$$q \le P_j, \quad \frac{P_{j-1}}{qN} < |\lambda| \le \frac{P_j}{qN}$$

or

$$P_{j-1} < q \le P_j, \quad |\lambda| \le \frac{P_j}{qN},$$

and therefore

$$P_{j-1}(q, a_i)^{-1} \ll q_i(1 + |a_i|\,|\lambda| N_i^3) \ll P_j,$$

where $q_i = q/(q, a_i)$. Hence Lemma 3.3 gives

$$S_i(\alpha) \ll q^{\varepsilon} N_i \{P_j^{1/2} N_i^{-1/2} + N_i^{-1/5} + (q, a_i)^{1/2} P_j^{-1/2}\} \log^c N_i.$$

Thus,

$$S_1(\alpha) \cdots S_9(\alpha) \ll q^{\varepsilon} L^c N_1 \cdots N_9 \prod_{i=1}^{9} \{P_j^{1/2} N_i^{-1/2} + N_i^{-1/5} + (q, a_i)^{1/2} P_j^{-1/2}\},$$

and hence

$$\begin{aligned}
&\int_{E(P_j)\setminus E(P_j)} |S_1(\alpha) \cdots S_9(\alpha)|\, d\alpha \\
&\quad \ll N_1 \cdots N_9 L^c \sum_{q \le P_j} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{P_j^{1+\varepsilon}}{qN} \prod_{i=1}^{9} \{P_j^{1/2} N_i^{-1/2} + N_i^{-1/5} + (q, a_i)^{1/2} P_j^{-1/2}\} \\
&\quad \ll \frac{N_1 \cdots N_9}{N} L^c \sum_{q \le P_j} P_j^{1+\varepsilon} \prod_{i=1}^{9} \{P_j^{1/2} N_i^{-1/2} + N_i^{-1/5} + (q, a_i)^{1/2} P_j^{-1/2}\},
\end{aligned}$$

https://doi.org/10.4153/CMB-2012-025-0 Published online by Cambridge University Press

where we used that the measure of $E(P_j)\setminus E(P_{j-1})$ does not exceed that of $E(P_j)$ and the measure of every interval of $E(P_j)$ is P_j/qN.

It therefore follows from (4.1) that

$$\begin{aligned}
&\int_{\mathfrak{M}\setminus\mathfrak{M}_0} |S_1(\alpha)\cdots S_9(\alpha)|\,d\alpha \\
&\ll \frac{N_1\cdots N_9}{N} L^c \sum_{j=1}^{h}\sum_{q\le P_j} P_j^{1+\varepsilon}\prod_{i=1}^{9}\{P_j^{1/2}N_i^{-1/2}+N_i^{-1/5}+(q,a_i)^{1/2}P_j^{-1/2}\} \\
&\ll \frac{N_1\cdots N_9}{N} L^c \sum_{j=1}^{h}\sum_{q\le P_j} P_j^{1+\varepsilon}\Big\{P_j^{9/2}\Big(\frac{N}{A}\Big)^{-3/2}+\Big(\frac{N}{A}\Big)^{-3/5}+(q,a_1\cdots a_9)P_j^{-9/2}\Big\} \\
&\ll \frac{N_1\cdots N_9}{N} L^c \sum_{j=1}^{h}\Big\{P_j^{13/2+\varepsilon}\Big(\frac{N}{A}\Big)^{-3/2}+P_j^{2+\varepsilon}\Big(\frac{N}{A}\Big)^{-3/5}+P_j^{-5/2}\sum_{q\le P_j}(q,a_1\cdots a_9)\Big\} \\
&\ll \frac{N_1\cdots N_9}{N} L^c h\Big\{P^{13/2+\varepsilon}\Big(\frac{N}{A}\Big)^{-3/2}+P^{2+\varepsilon}\Big(\frac{N}{A}\Big)^{-3/5}+P_0^{-3/2}A^{\varepsilon}\Big\} \\
&\ll \frac{N_1\cdots N_9}{NL},
\end{aligned}$$

where we used the symmetry of $a_1,\dots,a_9$, the elementary estimate

$$\sum_{q\le x}(q,b)\ll xb^{\varepsilon},$$

the definition of P and $h\ll L$, we see that

$$\int_{\mathfrak{M}\setminus\mathfrak{M}_0} |S_1(\alpha)\cdots S_9(\alpha)|\,d\alpha \ll \frac{N_1\cdots N_9}{NL}.$$

This proves Theorem 2.1.

5 The Proof of Main Theorems

Let N be a parameter with $N\ge A^{43+\varepsilon}$ that also satisfies hypothesis (i) or (ii) of Lemma 2.4 according as $a_1,\dots,a_9$ are all positive or not. In Section 4, we gave the asymptotic formula of the major arcs, and now we turn to the estimation of $\int_{\mathfrak{m}}$.

When $\alpha\in\mathfrak{m}$, there exist integers a and q satisfying (3.5) with $b=b_9$ and $X=N_9$ and such that $q+N|q\alpha-a|\ge P$. Obviously, P satisfies

$$2|b|N_9^{1/7}<P\le N_9.$$

We can apply Lemma 3.4 to get

$$\sup_{\alpha\in\mathfrak{m}}|S_9(\alpha)|\ll N_9^{13/14+\varepsilon}+N_9^{1+\varepsilon}|a_9|^{1/2}P^{-1/2}\ll N_9^{13/14+\varepsilon}.$$

We have the following mean-value estimate for $S_j(\alpha)$:

$$\int_0^1 |S_j(\alpha)|^8 \, d\alpha \ll L^8 \sum_{\substack{m_1^3+\cdots+m_4^3=m_5^3+\cdots+m_8^3 \\ m_\nu^3 \le N_j, \nu=1,\ldots,8}} 1 \ll N_j^{5/3+\varepsilon},$$

which in combination with Hölder's inequality gives

$$\int_0^1 |S_1(\alpha)\cdots S_8(\alpha)| \, d\alpha \ll \frac{N^{5/3+\varepsilon}}{|a_1\cdots a_8|^{5/24}}.$$

Therefore,

$$\int_{\mathfrak{m}} |S_1(\alpha)\cdots S_9(\alpha)| \, d\alpha \ll \frac{N_9^{13/14+\varepsilon}\cdot N^{5/3+\varepsilon}}{|a_1\cdots a_8|^{5/24}} \ll \frac{N^{83/42+\varepsilon}}{|a_1\cdots a_8|^{5/24}\,|a_9|^{13/42}}.$$

Thus,

$$r(n) = \frac{1}{3^9}\mathfrak{S}(n,P)\mathfrak{J}(n) + O\Big(\frac{N^2}{|a_1\cdots a_9|^{1/3}L}\Big) + O\Big(\frac{N^{83/42+\varepsilon}}{|a_1\cdots a_8|^{5/24}|a_9|^{13/42}}\Big).$$

If $n = N$ and all of $a_1, \ldots, a_9$ are positive, then

$$\frac{N^{83/42+\varepsilon}}{|a_1\cdots a_8|^{5/24}\,|a_9|^{13/42}} \ll \frac{N^2}{|a_1\cdots a_9|^{1/3}L},$$

provided that $N \gg A^{43+\varepsilon}$.

Thus,

$$r(n) \gg |a_1\cdots a_9|^{-1/3}N^2(\log\log N)^{-c}.$$

On the other hand, if not all of $a_1, \ldots, a_9$ are the same sign and $N \ge 27|n|$, then

$$a_1p_1^3 \le |n| + |a_2|p_2^3 + \cdots + |a_9|p_9^3 \le |n| + 8N,$$

or

$$a_1p_1^3 \ll |n| + A^{43+\varepsilon}.$$

Therefore, without any loss of generality, for all $1 \le j \le 9$, we have

$$p_j \ll |n|^{1/3} + A^{14+\varepsilon}.$$

This proves Theorems 1.1 and 1.2.

Acknowledgements The author would like to express his thanks to the referee for many useful suggestions and comments on the manuscript.

References

[1] A. Baker, *On some diophantine inequalities involving primes.* J. Reine Angew. Math. **228**(1967), 166–181.

[2] K. K. Choi, *A numerical bound for Baker's constant—some explicit estimates for small prime solutions of linear equations.* Bull. Hong Kong Math. Soc. **1**(1997), 1–19.

[3] K. K. Choi and A. V. Kumchev, *Mean values of Dirichlet polynomials and applications to linear equations with prime variables.* Acta Arith. **123**(2006), 125–142. http://dx.doi.org/10.4064/aa123-2-2

[4] ———, *Quadratic equations with five prime unknowns.* J. Number Theory **107**(2004), 357–367. http://dx.doi.org/10.1016/j.jnt.2004.03.007

[5] K. K. Choi and J. Y. Liu, *Small prime solutions of quadratic equations II.* Proc. Amer. Math. Soc. **133**(2005), 945–951. http://dx.doi.org/10.1090/S0002-9939-04-07784-6

[6] ———, *Small prime solutions of quadratic equations.* Canad. J. Math. **54**(2002), 71–91. http://dx.doi.org/10.4153/CJM-2002-004-4

[7] K. K. Choi, M. C. Liu, and K. M. Tsang, *Conditional bounds for small prime solutions of linear equations.* Manuscripta Math. **74**(1992), 321–340. http://dx.doi.org/10.1007/BF02567674

[8] G. Harman and A. V. Kumchev, *On sums of squares of primes.* Math. Proc. Cambridge Philos. Soc. **140**(2006), 1–13. http://dx.doi.org/10.1017/S0305004105008819

[9] L. K. Hua, *Some results in the additive prime number theory.* Quart. J. Math. (Oxford) **9**(1938), 68–80.

[10] A. V. Kumchev, *On Weyl sums over primes and almost primes.* Michigan Math. J. **54**(2006), 243–268. http://dx.doi.org/10.1307/mmj/1156345592

[11] D. Leung, *Small prime solutions to cubic Diophantine equations.* Master's thesis, Simon Fraser University, 2006.

[12] H. Z. Li, *Small prime solutions of some ternary linear equations.* Acta Arith. **98**(2001), 293–309. http://dx.doi.org/10.4064/aa98-3-6

[13] M. C. Liu and K. M. Tsang, *Small prime solutions of some additive equations.* Monatsh. Math. **111**(1991), 147–169. http://dx.doi.org/10.1007/BF01332353

[14] ———, *Small prime solutions of linear equations.* In: Théorie des nombres (Québec, PQ, 1987), de Gruyter, Berlin, 1989, 595–624.

[15] M. C. Liu and T. Z. Wang, *A numerical bound for small prime solutions of some ternary linear equations.* Acta Arith. **86**(1998), 343–383.

[16] J. Y. Liu, *Enlarged major arcs in additive problems, II.* Proc. Steklov Inst. Math. **276**(2012), 176–192.

[17] ———, *On Lagrange's theorem with prime variables.* Quart. J. Math. (Oxford) **54**(2003), 453–462. http://dx.doi.org/10.1093/qmath/hag028

[18] J. Y. Liu and K. M. Tsang, *Small prime solutions of ternary linear equations.* Acta Arith. **118**(2005), 79–100. http://dx.doi.org/10.4064/aa118-1-5

[19] X. M. Ren, *On exponential sums over primes and application in Waring–Goldbach problem.* Sci. China Ser. A **48**(2005), 785–797. http://dx.doi.org/10.1360/03ys0341

Department of Mathematics, School of Science, Tianjin University, Tianjin 300072, P. R. China
e-mail: zhixinliu@tju.edu.cn