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Li coefficients and the quadrilateral zeta function

Kajtaz H. Bllaca, Kamel Mazhouda and Takashi Nakamura

Abstract. **In this note, we study the Li coefficients $\lambda_{n,a}$ for the quadrilateral zeta function. Furthermore, we give an arithmetic and asymptotic formula for these coefficients. Especially, we show that for any fixed $n \in \mathbb{N}$, there exists $a > 0$ such that $\lambda_{2n-1,a} > 0$ and $\lambda_{2n,a} < 0$.**

1 Introduction and statement of main results

1.1 Li coefficients

The Riemann hypothesis is a critical question in analytic number theory. As such, it is interesting to examine different ways to frame it, which may shed more light on its resolution. In 1997, Xian-Jin Li has discovered a new positivity criterion for the Riemann hypothesis (RH). In [10] he defined the Li coefficients for the Riemann zeta function as

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}\left[s^{n-1}\log \xi(s)\right]_{s=1},$$

where ξ is the completed Riemann zeta function defined by

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

which satisfies $\xi(s) = \xi(1-s)$ and gave a simple equivalence criterion for the (RH): (RH) is true if and only if these coefficients are nonnegative for every positive integer n. The Li coefficients λ_n can be written as follows

$$\lambda_n = \sum_{\rho}^{*}\left[1-\left(1-\frac{1}{\rho}\right)^n\right] = \lim_{T\to\infty}\sum_{\rho;|\mathrm{Im}(\rho)|\le T}\left[1-\left(1-\frac{1}{\rho}\right)^n\right],$$

where the sum runs over the nontrivial zeros of the Riemann zeta function counted with multiplicity. This criterion is generalized by Bombieri and Lagarias [4] for any arbitrarily multiset of numbers assuming certain convergence conditions. Voros [19, section 3.3] has proved that the (RH) true is equivalent to the growth of λ_n as $\frac{1}{2}n\log n$ determined by its archimedean part, while the Riemann hypothesis false is equivalent to the oscillations of λ_n with exponentially growing amplitude, determined by its finite part. The Li coefficients were generalized in two ways; by generalizing these coefficients to various sets of functions (the Selberg class, the class of automorphic L-functions, zeta function on function fields,... [8, 11, 17]), and by introducing new parameter in its definition (see

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[12]). The Li coefficients (and its generalizations) has generated a lot of research interest due to its applicability and simplicity.

1.2 Quadrilateral zeta function

Recall the definitions of Hurwitz and periodic zeta functions. The Hurwitz zeta function $\zeta(s,a)$ is defined by the series

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \qquad \sigma > 1, \quad 0 < a \le 1.$$

The function $\zeta(s,a)$ is a meromorphic function with a simple pole at $s = 1$ whose residue is 1 (see for example [1, Section 12]). The periodic zeta function $\mathrm{Li}_s(e^{2\pi i a})$ is defined by

$$\mathrm{Li}_s(e^{2\pi i a}) := \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^s}, \qquad \sigma > 1, \quad 0 < a \le 1$$

(see for instance [1, Exercise 12.2]). Note that the function $\mathrm{Li}_s(e^{2\pi i a})$ with $0 < a < 1$ is analytically continuable to the whole complex plane since $\mathrm{Li}_s(e^{2\pi i a})$ does not have any pole, that is shown by the fact that the Dirichlet series of $\mathrm{Li}_s(e^{2\pi i a})$ converges uniformly in each compact subset of the half-plane $\sigma > 0$ when $0 < a < 1$ (see for example [9, p. 20]). For $0 < a \le 1/2$, we define zeta functions

$$Z(s,a) := \zeta(s,a) + \zeta(s,1-a), \qquad P(s,a) := \mathrm{Li}_s(e^{2\pi i a}) + \mathrm{Li}_s(e^{2\pi i(1-a)}),$$
$$2Q(s,a) := Z(s,a) + P(s,a), \qquad \xi_Q(s,a) := s(s-1)\pi^{-s/2}\Gamma(s/2)Q(s,a).$$

We can see that $Q(s,a)$ is meromorphic functions with a simple pole at $s = 1$. In addition, we have $Q(0,a) = -1/2 = \zeta(0)$ and $\xi_Q(s,a) = \xi_Q(1-s,a)$ which is proved by

$$Q(1-s,a) = \Gamma_{\cos}(s)Q(s,a), \qquad \Gamma_{\cos}(s) := \frac{2\Gamma(s)}{(2\pi)^s}\cos\left(\frac{\pi s}{2}\right) \tag{1.1}$$

(see [13, Theorem 1.1]). Moreover, the function $Q(s,a)$ has the following properties. When $a = 1/6, 1/4, 1/3$ and $1/2$, the Riemann hypothesis holds true if and only if all non real zeros of $Q(s,a)$ are on the line $\mathrm{Re}(s) = 1/2$ (see [15, Proposition 1.3]). Let $N_{\mathrm{Q}}^{\mathrm{CL}}(T)$ the number of the zeros of $Q(s,a)$ on the line segment from $1/2$ to $1/2 + iT$. In [13, Theorem 1.2], the third author proved that for any $0 < a \le 1/2$, there exist positive constants $A(a)$ and $T_0(a)$ such that

$$N_{\mathrm{Q}}^{\mathrm{CL}}(T) \ge A(a)T \quad \text{whenever} \quad T \ge T_0(a).$$

Next, let $N_F(T)$ count the number of non real zeros of a function $F(s)$ having $|\mathrm{Im}(s)| < T$. Then for any $0 < a \le 1/2$,

$$N_\zeta(T) - N_Q(T) = O_a(T),$$

and the third author [15, Proposition 1.8] proved that

$$N_Q(T) = \frac{T}{\pi}\log T - \frac{T}{\pi}\log(2e\pi a^2) + O_a(\log T).$$

Furthermore, he [15, Theorem 1.1] proved that there is a unique absolute $a_0 \in (0, 1/2)$ such that

$$Q(1/2, a) > 0 \iff 0 < a < a_0.$$

In addition, it is proved in [15, Corollary 1.2] that all real zeros of $Q(s, a)$ are simple and are located only at the negative even integers just like $\zeta(s)$ if and only if $a_0 < a \leq 1/2$. Let us note by Z_Q the set of all non-trivial zeros ρ_a of $\xi_Q(s, a)$. Since it is an entire function of order 1, one has

$$\xi_Q(s, a) = e^{A+Bs} \prod_{\rho_a \in Z_Q} \left(1 - \frac{s}{\rho_a}\right) e^{\frac{s}{\rho_a}} = \xi_Q(0, a) \prod_{\rho_a \in Z_Q} \left(1 - \frac{s}{\rho_a}\right), \tag{1.2}$$

where $e^A = 1/2$, $B = \frac{Q'}{Q}(0, a) - 1 - \frac{\gamma + \log \pi}{2}$ and γ denotes the Euler constant. Note that $Q'(0, a)$ is given explicitly in [15, Theorem 1.5].

1.3 Main results

Recall that $\zeta(1-s) = \Gamma_{\cos}(s)\zeta(s)$ and $Q(1-s, a) = \Gamma_{\cos}(s)Q(s, a)$ by (1.1). However, the function $Q(s, a)$ does not have an Euler product except for $a = 1/6, 1/4, 1/3$ and $1/2$. Hence, the function $Q(s, a)$ is a suitable object to consider the influence of not Riemann's functional equation but an Euler product to zeros of zeta functions. We show a criterion for non-vanishing of $Q(s, a)$ in terms of the positivity of the Li coefficients, an arithmetic and asymptotic formula for these coefficients in Theorems 1.1, 1.2 and 1.4, respectively. It should be emphasised that $\lambda_{n,a}$ defined in (1.3) are the first Li coefficients that we can explicitly give $n \in \mathbb{N}$ such that $\lambda_{n,a} < 0$. There is a possibility that this fact would give an idea to find negative Li coefficients for $\zeta(s)$ if they would exist.

For $n \neq 0$, the Li's coefficients attached to $Q(s, a)$ non vanishing at zero is defined by the sum

$$\lambda_{n,a} := \sum_{\rho_a \in Z_Q}^{*} \left(1 - \left(1 - \frac{1}{\rho_a}\right)^n\right) = \lim_{T \longmapsto \infty} \sum_{|\mathrm{Im}(\rho_a)| \leq T}^{*} \left(1 - \left(1 - \frac{1}{\rho_a}\right)^n\right).$$

The symmetry $\rho_a \longmapsto 1 - \rho_a$ in the set Z_Q of non-trivial zeros of $Q(s, a)$ implies that $\lambda_{-n,a} = \overline{\lambda_{n,a}} = \lambda_{n,a}$ for all $n \in \mathbb{N}$. So, $\lambda_{n,a}$ are real. We have also

$$\lambda_{n,a} := \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left[s^{n-1} \log \xi_Q(s, a)\right]_{s=1}. \tag{1.3}$$

Moreover, from (1.2) we have (see [4, Equations (2.3) and (2.4)] or [17, Appendix A])

$$\sum_{n=0}^{\infty} \lambda_{n+1,a} s^n = \frac{d}{ds} \log \left[\xi_Q\left(\frac{1}{1-s}, a\right)\right].$$

As an analogue of Li coefficients for the Riemann zeta function, we have the following.

Theorem 1.1 *The function $Q(s, a)$ does not vanish when* $\mathrm{Re}(s) > 1/2$ *if and only if $\lambda_{n,a} \geq 0$ for all $n \in \mathbb{N}$.*

An arithmetic formula for $\lambda_{n,a}$ is stated in the following theorems.

***Theorem* 1.2** *We have*

$$\lambda_{n,a} = 1 - \frac{n}{2}(\log(4\pi) + \gamma) + \sum_{k=2}^{n}(-1)^k \binom{n}{k} \left(1 - 2^{-k}\right) \zeta(k) + \sum_{k=1}^{n} \binom{n}{k} \gamma_Q(k-1),$$

where $\gamma_Q(n)$ *are defined as follows*

$$\frac{Q'}{Q}(s+1,a) + \frac{1}{s} = \sum_{n=0}^{\infty} \gamma_Q(n) s^n.$$

***Theorem* 1.3** *For* $a = 1/2, 1/3, 1/4, 1/6$, *under the RH we have*

$$\lambda_{n,a} = \frac{n}{2}\log n + \frac{n}{2}(\gamma - 1 - \log 2\pi) + O(\sqrt{n}\log n).$$

For a fixed $l \in \mathbb{N}$, we have the following asymptotic formula of $\lambda_{l,a}$ when $a \to +0$. We can see that there exists $n \in \mathbb{N}$ such that $\lambda_{n,a} < 0$ by Theorem 1.1 and the fact that $Q(s,a)$ does not satisfy an analogue of the Riemann hypothesis when $a \in \mathbb{Q} \cap (0, 1/2) \setminus \{1/6, 1/4, 1/3\}$ (see [15, Proposition 1.4]). Clearly, this argument gives no information on the frequency of $n \in \mathbb{N}$, the smallest $n \in \mathbb{N}$ such that $\lambda_{n,a} < 0$ and so on. However, the next theorem implies that $\lambda_{2n,a} < 0$ if we fix any $n \in \mathbb{N}$ and then we take $a > 0$ sufficiently small.

***Theorem* 1.4** *Fix* $l \in \mathbb{N}$. *Then it holds that*

$$\lambda_{l,a} = \frac{(-1)^{l+1}}{(2a)^l} + O_l\left(a^{1-l}|\log a|\right), \qquad a \to +0.$$

Especially, for any fixed $n \in \mathbb{N}$, *there are* $a > 0$ *such that*

$$\lambda_{2n-1,a} > 0 \quad \text{and} \quad \lambda_{2n,a} < 0.$$

2 Proofs

2.1 Proof of Theorem 1.1

Proof of Theorem 1.1 Since $\lambda_{-n,a} = \overline{\lambda_{n,a}} = \lambda_{n,a}$ for all $n \in \mathbb{N}$, then $\mathrm{Re}(\lambda_{-n,a}) = \mathrm{Re}(\lambda_{n,a}) = \lambda_{n,a}$. Using that $\xi_Q(s,a)$ is an entire function of order 1, and its zeros lie in the critical strip $0 < \mathrm{Re}(s) < 1$, we obtain that the series $\sum_{\rho \in Z_Q} \frac{1+|\mathrm{Re}(\rho)|}{(1+|\rho|)^2}$ is convergent. Application of [4, Theorem 1] to the multiset Z_Q of zeros of $Q(s,a)$ gives that $\mathrm{Re}(\rho) \le 1/2$ if and only if $\lambda_{n,a} \ge 0$ for all $n \in \mathbb{N}$. Now, the application of the same theorem to the multiset $1 - Z_Q = Z_Q$ gives $\mathrm{Re}(\rho) \ge 1/2$ if and only if $\lambda_{n,a} \ge 0$. This completes the proof.

Theorem 1.1 can be also proved by the same argument used in [5, Theorem 1] which is due to Oesterlé.

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2.2 Proof of Theorem 1.2

Proof of Theorem 1.2 From the expression of $\xi_Q(s,a)$, one has

$$\frac{\xi_Q'}{\xi_Q}(s,a) = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}(s/2) + \frac{Q'}{Q}(s,a)$$

which is rewritten as

$$\frac{\xi_Q'}{\xi_Q}(s+1,a) = \frac{1}{s+1} + \frac{1}{s} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}((s+1)/2) + \frac{Q'}{Q}(s+1,a). \tag{2.1}$$

Note that $Q(s,a)$ is a meromorphic function on the whole complex plane, which is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1 (see [13, Section 2.1]). Let define the coefficients $\gamma_Q(n)$ and $\tau_Q(n)$ as follows

$$\frac{Q'}{Q}(s+1,a) + \frac{1}{s} = \sum_{n=0}^{\infty} \gamma_Q(n)s^n \tag{2.2}$$

and

$$-\frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}((s+1)/2) = \sum_{n=0}^{\infty} \tau_Q(n)s^n. \tag{2.3}$$

By Equation (1.2), one has

$$\log\xi_Q(s,a) = \log\xi_Q(0,a) - \sum_{\rho_a\in Z_Q}\sum_{m=1}^{\infty}\frac{1}{m\rho^m}s^m.$$

From the functional equation for the function $\xi_Q(s,a)$, in the neighborhood of $s = 0$, we have

$$\frac{\xi_Q'}{\xi_Q}(s+1,a) = -\frac{\xi_Q'}{\xi_Q}(-s,a) = \sum_{m=0}^{\infty}(-1)^m\sum_{\rho_a\in Z_Q}\frac{1}{\rho^{m+1}}s^m. \tag{2.4}$$

Comparing Equations (2.1), (2.2), (2.3) and (2.4), we get

$$(-1)^m\sum_{\rho_a\in Z_Q}\frac{1}{\rho^{m+1}} = (-1)^m + \gamma_Q(m) + \tau_Q(m),$$

for $m \geq 0$. Hence, the definition of $\lambda_{n,a}$ yields

$$\lambda_{n,a} = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{\rho_a\in Z_Q}\frac{1}{\rho^k} = 1 + \sum_{k=1}^{n}\binom{n}{k}\gamma_Q(k-1) + \sum_{k=1}^{n}\binom{n}{k}\tau_Q(k-1),$$

where

$$\tau_Q(0) = -\frac{1}{2}\log\pi + \frac{1}{2}\psi(1/2) \text{ and } \tau_Q(k-1) = (-1)^k\sum_{m=0}^{\infty}\frac{1}{(2m+1)^k}$$

using that $\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \frac{z}{k(k+z)}$. Here $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$ is the logarithmic derivative of the Gamma function. Since $\psi(1/2) = -\gamma - 2\log 2$, we obtain

$$\begin{aligned} \lambda_{n,a} &= 1 - \frac{n}{2}(\log(4\pi) + \gamma) + \sum_{k=2}^{n} (-1)^k \binom{n}{k} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^k} + \sum_{k=1}^{n} \binom{n}{k} \gamma_Q(k-1), \\ &= 1 - \frac{n}{2}(\log(4\pi) + \gamma) + \sum_{k=2}^{n} (-1)^k \binom{n}{k} \left(1 - 2^{-k}\right) \zeta(k) + \sum_{k=1}^{n} \binom{n}{k} \gamma_Q(k-1). \end{aligned}$$

The equality above implies Theorem 1.2.

2.3 Proof of Theorem 1.3

Proof of Theorem 1.3 Let us note that

$$\sum_{k=2}^{n} (-1)^k \binom{n}{k} \left(1 - 2^{-k}\right) \zeta(k) = \sum_{k=2}^{n} (-1)^l \binom{n}{k} \frac{\zeta(k, 1/2)}{2^k},$$

where $\zeta(s, a)$ is the Hurwitz zeta function defined in Section 1.2. With notation of Flajolet and Vespas [7, Lines 2-4 page 70], this is $A_n(1, 2)$ and which equal to

$$\frac{n}{2}\psi(n) + n\left(\gamma - \frac{1}{2} + \frac{1}{2}\log 2\right) + o(1),$$

where the o(1) error term above is exponentially small and oscillating and equal to

$$\frac{1}{2}\left(\frac{n}{\pi}\right)^{1/4} \exp(-\sqrt{2\pi n}) \cos\left(\sqrt{2\pi n} - \frac{5\pi}{8}\right) + O\left(n^{-1/4} e^{-\sqrt{2\pi n}}\right).$$

Then we have

$$\lambda_{n,a} = \frac{n}{2}\log n + \frac{n}{2}(\gamma - 1 - \log 2\pi) + \sum_{k=1}^{n} \binom{n}{k} \gamma_Q(k-1) + O(1).$$

It remain to prove that

$$\sum_{k=1}^{n} \binom{n}{k} \gamma_Q(k-1) = O(\sqrt{n}\log n). \tag{2.5}$$

To do so, we follows very closely the lines of the proof of the corresponding result in [8, Theorem 6.1] or [16, Lemma 3.3] and it will be shortened. We use the following kernel function

$$k_n(s) := \left(1 + \frac{1}{s}\right)^n - 1 = \sum_{k=1}^{n} \binom{n}{k} \frac{1}{s^k}.$$

The residue theorem gives

$$\sum_{k=1}^{n} \binom{n}{k} \gamma_Q(k-1) = \frac{1}{2i\pi} \int_C k_n(s) \left(-\frac{Q'}{Q}(s+1, a)\right) ds,$$

where C is a contour enclosing the point $s = 0$ counterclockwise on a circle of small enough positive radius. The residue comes entirely from the singularity at $s = 0$, as no

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other singularities lie inside the contour. Let $T = \sqrt{n}+\epsilon_n$, for some $0 < \epsilon_n < 1$. Now we follow very closely the lines in [16, p. 1106 and p. 1107] using that the function $\frac{Q'}{Q}(s,a)$ satisfies the properties *

$$\frac{Q'}{Q}(s,a) = \sum_{\rho_a;\ |\mathrm{Im}(\rho_a - s)|<1} \frac{1}{s-\rho_a} + O(\log(1+|s|)),$$

for $-2 < \mathrm{Re}(s) < 2$ and

$$\left|\frac{Q'}{Q}(s+1,a)\right| = O(\log^2 T),$$

for $-2 \leq \mathrm{Re}(s) \leq 2$, and we get

$$\sum_{k=1}^{n} \binom{n}{k} \gamma_Q(k-1) = \lambda_{-n,a,T} + O(\sqrt{n}\log n),$$

where

$$\lambda_{-n,a,T} = \sum_{\rho_a \in Z_Q; |Im(\rho_a| \leq T}^{*} \left(1 - \left(1 - \frac{1}{\rho_a}\right)^n\right),$$

with $T = \sqrt{n} + \epsilon_n$. For $a = 1/2, 1/3, 1/4, 1/6$, under the RH, since $\left|1 - \frac{1}{\rho_a}\right| = 1$ and using formula of $N_Q(T)$ given in Section 1.2, we obtain $\lambda_{n,a,T} = O(T\log T + 1)$. Therefore, equation (2.5) follows from that $\lambda_{-n,a,\sqrt{n}} = \lambda_{-n,a,\sqrt{n}} = O(\sqrt{n}\log n)$.

Remark. Since $2Q(s,a) := Z(s,a) + P(s,a)$, from Corollary 2.3 below and [6, Equation (1.18)], we obtain

$$\gamma_Q(n) = \frac{1}{2}\left(\delta_n(a) + \frac{(-1)^n}{n!}(l_n(a) + l_n(1-a))\right),$$

where $\delta_n(a) = \frac{|\log a|^n}{an!} + O(1)$ and $l_n(a)$ are the coefficients in the expansion of $\mathrm{Li}_s(e^{2\pi i a})$ at $s = 1$; for $a \notin \mathbb{Z}$ one has

$$\mathrm{Li}_s(e^{2\pi i a}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} l_n(a)(s-1)^n.$$

2.4 Proof of Theorem 1.4

To show Theorem 1.4, we quote the following lemmas from [2] and [3].

***Lemma* 2.1** *([3, Theorem 1])* *We set*

$$(s-1)\zeta(s,a) = 1 + \sum_{n=0}^{\infty} \gamma_n(a)(s-1)^{n+1}, \qquad 0 < a \leq 1.$$

---

*These properties are well known for the Riemann zeta-function. The proof for the function $Q(s,a)$ is exactly the same since the Riemann-von Mangoldt formula holds for $Q(s,a)$ (see [15, Proposition 2.5] or [18, Page 217]).

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Then it holds that

$$\gamma_n(a) = \frac{(-1)^n}{n!} \lim_{m\to\infty}\left(\sum_{k=0}^{m} \frac{\log^n(k+a)}{k+a} - \frac{\log^{n+1}(m+a)}{n+1}\right).$$

***Lemma* 2.2** *([2, (26)])* *Let $0 < a \le 1$ and n be a non-negative integer. Then one has*

$$\zeta^{(n)}(0,a) = \left(\frac{1}{2} - a\right)|\log a|^n - n! + n!a \sum_{m=n}^{\infty} \frac{|\log a|^m}{m!}$$
$$+ (-1)^n n \int_0^\infty \frac{\varphi(x)\log^{n-1}(x+a)}{(x+a)^2}dx - (-1)^n n(n-1) \int_0^\infty \frac{\varphi(x)\log^{n-2}(x+a)}{(x+a)^2}dx,$$

where $\varphi(x) = \int_0^x (y - \lfloor y \rfloor - 1/2)dy$ is periodic with period 1 and satisfies $2\varphi(x) = x(x-1)$ if $0 \le x \le 1$.

By using the Lemmas above, we immediately obtain the following.

***Corollary* 2.3** *When $a > 0$ is sufficiently small,*

$$(s-1)Z(s,a) = 2 + \sum_{n=0}^{\infty} \delta_n(a)(s-1)^{n+1}, \qquad \delta_n(a) = \frac{|\log a|^n}{an!} + O(1),$$
$$Z(s,a) = \sum_{n=1}^{\infty} \epsilon_n(a)s^n, \qquad \epsilon_n(a) = O(|\log a|^n).$$

Proof The first formula and estimation are easily proved by Lemma 2.1 (see also [3, Theorem 2]). For the first integral in the Lemma 2.2, one has

$$\int_0^1 \frac{\varphi(x)\log^{n-1}(x+a)}{(x+a)^2}dx \ll \int_0^1 \frac{\log^{n-1}(x+a)}{x+a}dx = O(|\log a|^n),$$
$$\int_1^\infty \frac{\varphi(x)\log^{n-1}(x+a)}{(x+a)^2}dx \ll \int_1^\infty \frac{\log^{n-1}(x+a)}{(x+a)^2}dx = O(1)$$

from $x < x + a$ when $x, a > 0$. In addition, we have

$$a\sum_{m=n}^{\infty} \frac{|\log a|^m}{m!} \le a\sum_{m=0}^{\infty} \frac{|\log a|^m}{m!} = ae^{|\log a|} = ae^{-\log a} = 1, \qquad 0 < a < 1/2.$$

Hence, we obtain

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{\zeta^{(n)}(0,a)}{n!}s^n, \qquad \zeta^{(n)}(0,a) = O(|\log a|^n).$$

Therefore, we have $\epsilon_n(a) = O(|\log a|^n)$ and the second formula in this corollary by the definition of $Z(s,a)$ and $Z(0,a) = \zeta(0,a) + \zeta(0,1-a) = 0$ (see [14, (4.11)]). ■

Proof of Theorem 1.4 Recall the functional equation

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$$Z(1-s,a)=\Gamma_{\cos}(s)P(s,a), \qquad \Gamma_{\cos}(s):=\frac{2\Gamma(s)}{(2\pi)^s}\cos\Big(\frac{\pi s}{2}\Big)$$

(see [14, Lemma 4.11]). By using $\Gamma_{\cos}(s)\Gamma_{\cos}(1-s)=1$, we have

$$2Q(s,a)=Z(s,a)+P(s,a)=Z(s,a)+\Gamma_{\cos}(1-s)Z(1-s,a).$$

Let $|s-1|$ be sufficiently small. Then by $\lim_{s\to1}(s-1)Q(s,a)=1$, the equation above and the definitions of $Q(s,a)$ and $\xi_Q(s,a)$, we have

$$\begin{aligned}
&\frac{d^l}{ds^l}\big[s^{l-1}\log\xi_Q(s,a)\big]_{s=1}=\frac{d^l}{ds^l}\Big[s^{l-1}\log\big((s-1)Q(s,a)\big)+s^{l-1}\log\big(s\pi^{-s/2}\Gamma(s/2)\big)\Big]_{s=1}\\
&=\frac{d^l}{ds^l}\left[s^{l-1}\log\left(\frac{s-1}{2}\Big(Z(s,a)+\Gamma_{\cos}(1-s)Z(1-s,a)\Big)\right)\right]_{s=1}+O_l(1)\\
&=\frac{d^l}{ds^l}\left[s^{l-1}\log\left(1+\sum_{n=0}^{\infty}\Big(\delta'_n(a)+\epsilon'_n(a)\Big)(s-1)^{n+1}\right)\right]_{s=1}+O_l(1),
\end{aligned}$$

where $\delta'_n(a)$ and $\epsilon'_n(a)$ are defined by

$$\delta'_n(a):=\frac{\delta_n(a)}{2}, \qquad (s-1)\Gamma_{\cos}(1-s)Z(1-s,a)=2\sum_{n=0}^{\infty}\epsilon'_n(a)(s-1)^{n+1}.$$

Clearly, the second estimation in Corollary 2.3 implies

$$Z(1-s,a)=\sum_{n=1}^{\infty}\epsilon_n(a)(1-s)^n, \qquad \epsilon_n(a)=O(|\log a|^n).$$

Thus we can see that $\epsilon'_n(a)=O(|\log a|^{n+1})$ from $\lim_{s\to1}(s-1)\Gamma_{\cos}(1-s)=-2$ and the fact that the function $(s-1)\Gamma_{\cos}(1-s)$ does not depend on a. Put $\eta_n(a):=\delta'_n(a)+\epsilon'_n(a)$. Then, for $n\geq 0$, we have

$$\eta_n(a)=\frac{1}{n!}\frac{|\log a|^n}{2a}+O(|\log a|^{n+1}), \qquad a\to+0 \tag{2.6}$$

by Corollary 2.3. By virtue of

$$\begin{aligned}
(a_0x+a_1x^2+a_2x^3+\cdots)^m&=a_0^mx^m+\binom{m}{1}a_0^{m-1}a_1x^{m+1}+\cdots\\
(a_0x+a_1x^2+a_2x^3+\cdots)^{m-1}&=a_0^mx^{m-1}+\binom{m-1}{1}a_0^{m-2}a_1x^m+\cdots\\
&\ \ \vdots\\
(a_0x+a_1x^2+a_2x^3+\cdots)^1&=\cdots+a_mx^m+\cdots,
\end{aligned}$$

where $m\in\mathbb{N}$ and $a_m,x\in\mathbb{C}$, the coefficient of $(s-1)^l$ in the function

$$f(s,a):=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\left(\sum_{n=0}^{\infty}\eta_n(a)(s-1)^{n+1}\right)^m$$

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is expressed as

$$\frac{(-1)^{l+1}}{l}\big(\eta_0(a)\big)^l + \frac{(-1)^l}{l-1}\binom{l-1}{1}\eta_0(a)^{l-2}\eta_1(a) + \cdots + \frac{(-1)^{1+1}}{1}\eta_{l-1}(a). \qquad (2.7)$$

Note that the function above is estimated by

$$\frac{(-1)^{l+1}}{l}\big(\eta_0(a)\big)^l + O_l\big(\eta_0(a)^{l-2}\eta_1(a)\big) = \frac{(-1)^{l+1}}{l}(2a)^{-l} + O_l\big(a^{1-l}|\log a|\big) \qquad (2.8)$$

from (2.6) when $a \to +0$. We can find that

$$(s-1)\Big(Z(s,a) + \Gamma_{\cos}(1-s)Z(1-s,a)\Big) = 1 + \sum_{n=0}^{\infty}\eta_n(a)(s-1)^{n+1}$$

is analytic when $|s-1| < 1$ form the poles of $Z(s,a)$ and $\Gamma_{\cos}(1-s)$. So we can choose $|s-1| > 0$ such that

$$\sum_{n=0}^{\infty}\big|\eta_n(a)\big||s-1|^{n+1} < \frac{1}{2}.$$

Then, from (2.7), the Leibniz product rule, the definition of $\eta_n(a)$, and the Taylor expansion of $\log(1+x)$ with $|x| < 1$, one has

$$\begin{aligned}
\frac{d^l}{ds^l}\Big[s^{l-1}\log\xi_Q(s,a)\Big]_{s=1} &= \frac{d^l}{ds^l}\Big[s^{l-1}f(s,a)\Big]_{s=1} + O_l(1) \\
&= \binom{l}{l}\frac{(-1)^{l+1}}{l}l!\big(\eta_0(a)\big)^l + O_l\big(\eta_0(a)^{l-2}\eta_1(a)\big) && (\flat) \\
&\quad + \binom{l}{l-1}(l-1)\frac{(-1)^l}{l-1}(l-1)!\big(\eta_0(a)\big)^{l-1} + O_l\big(\eta_0(a)^{l-3}\eta_1(a)\big) && (\natural) \\
&\quad + \cdots + \binom{l}{1}(l-1)!\frac{(-1)^{1+1}}{1}\big(\eta_0(a)\big)^1 + O_l(1). && (\sharp)
\end{aligned}$$

Note that ($\flat$) comes from $f^{(l)}(s,a)$, ($\natural$) is deduced by $f^{(l-1)}(s,a)$, and ($\sharp$) derives from $f^{(1)}(s,a)$, $f^{(0)}(s,a)$ and $O_l(1)$ in the left-hand side of the formula above. Therefore, by (2.8), we obtain

$$\begin{aligned}
\frac{d^l}{ds^l}\Big[s^{l-1}\log\xi_Q(s,a)\Big]_{s=1} &= (-1)^{l+1}(l-1)!\big(\eta_0(a)\big)^l + O_l\big(\eta_0(a)^{l-2}\eta_1(a)\big) \\
&= (-1)^{l+1}\frac{(l-1)!}{(2a)^l} + O_l\big(a^{1-l}|\log a|\big)
\end{aligned}$$

which implies Theorem 1.4.

At the end of the paper, we give numerical computation for $\lambda_{n,a}$ by Mathematica 13.0. Let

$$\lambda_{n,a}^{[k]} := \frac{1}{(n-1)!}\frac{d^n}{ds^n}\big[s^{n-1}\log\xi_Q(s,a)\big]_{s=1-10^{-k}}, \qquad \lambda_{n,a}^{*} := \frac{(-1)^{n+1}}{(2a)^n}.$$

Then, we have the following:

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For $n = 1$, we have

$$
\begin{array}{lll}
a := 2^{-17} & \lambda_{1,a}^{[10]} = 65537\ldots & \lambda_{1,a}^{[10]}/\lambda_{1,a}^{*} = 1.00001\ldots \\
a := 2^{-18} & \lambda_{1,a}^{[10]} = 131074\ldots & \lambda_{1,a}^{[10]}/\lambda_{1,a}^{*} = 1.00002\ldots \\
a := 2^{-19} & \lambda_{1,a}^{[10]} = 262151\ldots & \lambda_{1,a}^{[10]}/\lambda_{1,a}^{*} = 1.00003\ldots \\
a := 2^{-17} & \lambda_{1,a}^{[11]} = 65536.6\ldots & \lambda_{1,a}^{[11]}/\lambda_{1,a}^{*} = 1.00001\ldots \\
a := 2^{-18} & \lambda_{1,a}^{[11]} = 131073\ldots & \lambda_{1,a}^{[11]}/\lambda_{1,a}^{*} = 1.00001\ldots \\
a := 2^{-19} & \lambda_{1,a}^{[11]} = 262145\ldots & \lambda_{1,a}^{[11]}/\lambda_{1,a}^{*} = 1.00000\ldots \\
a := 2^{-17} & \lambda_{1,a}^{[12]} = 655365\ldots & \lambda_{1,a}^{[12]}/\lambda_{1,a}^{*} = 1.00001\ldots \\
a := 2^{-18} & \lambda_{1,a}^{[12]} = 131073\ldots & \lambda_{1,a}^{[12]}/\lambda_{1,a}^{*} = 1.00000\ldots \\
a := 2^{-19} & \lambda_{1,a}^{[12]} = 262145\ldots & \lambda_{1,a}^{[12]}/\lambda_{1,a}^{*} = 1.00000\ldots
\end{array}
$$

For $n = 2$, we have

$$
\begin{array}{lll}
a := 2^{-17} & \lambda_{2,a}^{[10]} = -4.29352\ldots \times 10^{9} & \lambda_{2,a}^{[10]}/\lambda_{2,a}^{*} = 0.999663\ldots \\
a := 2^{-18} & \lambda_{2,a}^{[10]} = -1.7177\ldots \times 10^{10} & \lambda_{2,a}^{[10]}/\lambda_{2,a}^{*} = 0.999836\ldots \\
a := 2^{-19} & \lambda_{2,a}^{[10]} = -6.87162\ldots \times 10^{10} & \lambda_{2,a}^{[10]}/\lambda_{2,a}^{*} = 0.999952\ldots \\
a := 2^{-17} & \lambda_{2,a}^{[11]} = -4.29478\ldots \times 10^{9} & \lambda_{2,a}^{[11]}/\lambda_{2,a}^{*} = 0.999956\ldots \\
a := 2^{-18} & \lambda_{2,a}^{[11]} = -1.71753\ldots \times 10^{10} & \lambda_{2,a}^{[11]}/\lambda_{2,a}^{*} = 0.999736\ldots \\
a := 2^{-19} & \lambda_{2,a}^{[11]} = -6.87149\ldots \times 10^{10} & \lambda_{2,a}^{[11]}/\lambda_{2,a}^{*} = 0.999933\ldots \\
a := 2^{-17} & \lambda_{2,a}^{[12]} = -4.29477\ldots \times 10^{9} & \lambda_{2,a}^{[12]}/\lambda_{2,a}^{*} = 0.999955\ldots \\
a := 2^{-18} & \lambda_{2,a}^{[12]} = -1.6911\ldots \times 10^{10} & \lambda_{2,a}^{[12]}/\lambda_{2,a}^{*} = 0.984353\ldots \\
a := 2^{-19} & \lambda_{2,a}^{[12]} = -6.87187\ldots \times 10^{10} & \lambda_{2,a}^{[12]}/\lambda_{2,a}^{*} = 0.999989\ldots
\end{array}
$$

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Department of Mathematics, University of Prishtina, Mother Theresa, no. 5, Kosovo, 10000 Prishtina
e-mail: kajtaz.bllaca@uni-pr.edu.

Higher Institute of Applied Sciences and Technology, University of Sousse, Tunisia, 4003 Sousse and INSA Hauts-De-France, Univ. Polytechnique Hauts-De-France, FR CNRS 2037, CERAMATHS, France, F-59313 Valenciennes
e-mail: kamel.mazhouda@fsm.rnu.tn.

Department of Liberal Arts, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Japan, Noda-shi, Chiba-ken, 278-8510
e-mail: nakamuratakashi@rs.tus.ac.jp.

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