A drift-kinetic analytical model for scrape-off layer plasma dynamics at arbitrary collisionality

R. Jorge^{1,2,†}, P. Ricci¹ and N. F. Loureiro³

¹École Polytechnique Fédérale de Lausanne (EPFL), Swiss Plasma Center (SPC), CH-1015 Lausanne, Switzerland

²Instituto de Plasmas e Fusão Nuclear, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisboa, Portugal

³Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

(Received 4 September 2017; revised 17 October 2017; accepted 19 October 2017)

A drift-kinetic model to describe the plasma dynamics in the scrape-off layer region of tokamak devices at arbitrary collisionality is derived. Our formulation is based on a gyroaveraged Lagrangian description of the charged particle motion, and the corresponding drift-kinetic Boltzmann equation that includes a full Coulomb collision operator. Using a Hermite–Laguerre velocity space decomposition of the gyroaveraged distribution function, a set of equations to evolve the coefficients of the expansion is presented. By evaluating explicitly the moments of the Coulomb collision operator, distribution functions arbitrarily far from equilibrium can be studied at arbitrary collisionalities. A fluid closure in the high-collisionality limit is presented, and the corresponding fluid equations are compared with previously derived fluid models.

Key words: fusion plasma, plasma confinement, plasma nonlinear phenomena

1. Introduction

The success of the magnetic confinement fusion program relies on our ability to predict the dynamics of the plasma in the tokamak scrape-off layer (SOL). In this region, the plasma is turbulent with a fluctuation level of order unity (Ritz *et al.* 1987; Wootton *et al.* 1990; Hidalgo *et al.* 2002; LaBombard *et al.* 2005; D’Ippolito, Myra & Zweben 2011). The fluctuations are characterized by frequencies lower than the ion gyrofrequency (Endler *et al.* 1995; Agostini *et al.* 2011; Carralero *et al.* 2014; Garcia, Horacek & Pitts 2015), and the turbulent eddies, which include coherent radial propagation of filamentary structures (D’Ippolito, Myra & Krasheninnikov 2002; Carreras 2005; Seriani *et al.* 2007; D’Ippolito *et al.* 2011), have a radial extension comparable to the time-averaged SOL pressure gradient length L_p (Zweben *et al.* 2007).

In recent years, there has been a significant development of first-principles simulations of the SOL dynamics with both kinetic (Tskhakaya 2012) and gyrokinetic (Xu *et al.* 2007; Shi, Hakim & Hammett 2015; Chang *et al.* 2017; Shi *et al.* 2017) codes. However, as kinetic simulations of the SOL and edge regions remain prohibitive as they still are computationally extremely expensive, the less demanding fluid

† Email address for correspondence: rogerio.jorge@epfl.ch

simulations are the standard of reference (Dudson *et al.* 2009; Tamain *et al.* 2009; Easy *et al.* 2014; Halpern *et al.* 2016; Madsen *et al.* 2016). The fluid simulations are usually based on the drift-reduced Braginskii (Braginskii 1965; Zeiler, Drake & Rogers 1997) or gyrofluid (Ribeiro & Scott 2008; Held *et al.* 2016) models to evolve plasma density, fluid velocity and temperature. Fluid models assume that the distribution function is close to a local Maxwellian, and that scale lengths along the magnetic field are longer than the mean free path. However, kinetic simulations show that the plasma distribution function is far from Maxwellian in the SOL region (Lönneroth *et al.* 2006; Tskhakaya *et al.* 2008; Battaglia *et al.* 2014) and that collisionless effects in the SOL might become important (Batishchev *et al.* 1997). This is expected to be particularly true in ITER and other future devices that will be operated in the high confinement mode (H-mode) regime (Martin, Takizuka & Group 2008). In such cases, a transport barrier is formed that creates a steep pressure gradient at the plasma edge. If the pressure gradient exceeds a threshold value, edge-localized modes (ELMs) are destabilized (Leonard 2014), expelling large amounts of heat and particles to the wall. Describing structures with such high temperatures (and therefore low collisionality) with respect to the background SOL plasma requires therefore a model that allows for the treatment of arbitrary collision frequencies. Higher moments of the distribution function are needed for a proper SOL description (Hazeltine 1998).

Leveraging the development of previous models (Hammett *et al.* 1993; Beer & Hammett 1996; Sugama, Watanabe & Horton 2001; Ji & Held 2010; Zocco & Schekochihin 2011; Hatch *et al.* 2016; Hirvijoki *et al.* 2016; Parker 2016; Schekochihin *et al.* 2016; Mandell, Dorland & Landreman 2017), we construct here a moment hierarchy to evolve the SOL plasma dynamics. Our model is valid in arbitrary magnetic field geometries and, making use of the full Coulomb collision operator, at arbitrary collision frequencies. The model is derived within a full-F framework, as the amplitude of the background and fluctuating components of the plasma parameters in the SOL have comparable amplitude. We work within the drift approximation (Hinton & Hazeltine 1976; Cary & Brizard 2009), which assumes that plasma quantities have typical frequencies that are small compared to the ion gyrofrequency $\Omega_i = eB/m_i$, and their perpendicular spatial scale is small compared to the ion sound Larmor radius $\rho_s = c_s/\Omega_i$, with $c_s^2 = T_e/m_i$, T_e the electron temperature, B the magnitude of the magnetic field, e the electron charge, and m_i the ion mass. Moreover, we consider a Braginskii ordering (Braginskii 1965), where the species flow velocities are comparable to the ion thermal speed, as opposed to the drift ordering introduced by Mikhailovskii & Tsypin (1971) and extended and corrected by Catto & Simakov (2004), where flow velocities are comparable to the diamagnetic drift velocities.

More precisely, denoting $k_\perp \sim |\nabla_\perp \log \phi| \sim |\nabla_\perp \log n| \sim |\nabla_\perp \log T_e|$ and $\omega \sim |\partial_t \log \phi| \sim |\partial_t \log n| \sim |\partial_t \log T_e|$, with ϕ the electrostatic potential and n the plasma density, we introduce the ordering parameter ϵ such that

$$\epsilon \sim k_\perp \rho_s \sim \omega/\Omega_i \ll 1. \quad (1.1)$$

On the other hand, we let $k_\perp L_p \sim 1$ since turbulent eddies are observed to have an extension comparable to the scale lengths of the time-averaged quantities. These assumptions are in agreement with experimental measurements of SOL plasmas (LaBombard *et al.* 2001; Zweben *et al.* 2004; Myra *et al.* 2013; Carralero *et al.* 2014). We also order the electron collision frequency ν_{ei} as

$$\frac{\nu_{ei}}{\Omega_i} \sim \epsilon_v \lesssim \epsilon. \quad (1.2)$$

In addition, the ion collision frequency $\nu_i = \nu_{ii}$ is ordered as $\nu_{ii} \lesssim \epsilon^2 \Omega_i$ that, noticing $\nu_i \sim \sqrt{m_e/m_i} (T_e/T_i)^{3/2} \nu_e$ (with $\nu_e = \nu_{ei}$), yields

$$\left(\frac{\epsilon_v}{\epsilon^2}\right)^{2/3} \left(\frac{m_e}{m_i}\right)^{1/3} \lesssim \frac{T_i}{T_e} \lesssim 1. \quad (1.3)$$

The ordering in (1.3) can be used to justify applying our model in the cold ion limit, $T_i \ll T_e$, but allows for $T_i \sim T_e$. We note that in the SOL the ratio T_i/T_e is in the range $1 \lesssim T_i/T_e \lesssim 4$ (Kočan *et al.* 2011). The ion temperature in this range of values is seen to play a negligible role in determining the SOL turbulent dynamics, usually due to a steeper electron temperature profile compared with the ion one, which is usually below the threshold limit of the ion temperature gradient instability (Mosetto *et al.* 2015).

The ordering in (1.1)–(1.3) is justified in a wide variety of experimental conditions. For example, for a typical JET discharge (Erents *et al.* 2000; Liang *et al.* 2007; Xu *et al.* 2009) with the SOL parameters $B_T = 2.5$ T, $T_e \sim T_i \sim 20$ eV, $n_e \simeq 10^{19} \text{ m}^{-3}$ and $k_\perp \sim 1 \text{ cm}^{-1}$, we obtain $\epsilon_v \sim 0.016$ and $\epsilon \sim 0.0182$. For a medium-sized tokamak such as TCV (Rossel *et al.* 2012; Nespoli *et al.* 2017), estimating $B_T = 1.5$ T, $T_e \sim T_i \sim 40$ eV, $n_e \simeq 6 \times 10^{18}$ and $k_\perp \sim 1 \text{ cm}^{-1}$, we obtain $\epsilon_v \sim 6.2 \times 10^{-3}$ and $\epsilon \sim 0.043$. Finally, for small-sized tokamaks such as ISTTOK (Silva *et al.* 2011; Jorge *et al.* 2016), with $B_T = 0.5$ T, $T_e \sim T_i \sim 20$ eV, $n_e \simeq 0.8 \times 10^{18}$ and $k_\perp \sim 1 \text{ cm}^{-1}$, we obtain $\epsilon_v \sim 0.0072$ and $\epsilon \sim 0.091$. Lower values of ϵ_v , as in the presence of ELMs where temperatures can reach up to 100 eV (Pitts *et al.* 2003), are also included in the ordering considered here.

Following typical SOL experimental measurements (see, e.g. Zweben *et al.* (2007), Terry *et al.* (2009), Grulke *et al.* (2014)), we order $k_\parallel \sim 1/L_B \sim 1/R$, with L_B the background magnetic field spatial gradient scale and R the tokamak major radius, and take $k_\parallel \rho_s \sim \epsilon^3$. This yields

$$\frac{k_\parallel}{k_\perp} \sim \epsilon^2, \quad (1.4)$$

a lower ratio than the ones used in most drift-kinetic and gyrokinetic deductions (Hahn 1988; Hazeltine & Meiss 2003; Abel *et al.* 2013). The orderings in (1.2) and (1.4) imply that

$$k_\parallel \lambda_{mfp} \sim \sqrt{\frac{m_i}{m_e}} \frac{\epsilon^3}{\epsilon_v}, \quad (1.5)$$

which includes both the collisional regime $k_\parallel \lambda_{mfp} \lesssim 1$, when $\epsilon_v \sim \epsilon$, and the collisionless regime $k_\parallel \lambda_{mfp} \gg 1$, when $\epsilon_v \ll \epsilon$. Finally, the plasma parameter $\beta = nT_e/(B^2/2\mu_0)$ is ordered as $\beta \lesssim \epsilon^3$, implying that our equations describe plasma dynamics in an electrostatic regime.

Our model describes the evolution of the moments of the drift-kinetic Boltzmann equation at order ϵ^2 , taking into account the effect of collisions through a full Coulomb collision operator. The kinetic equation is based on a Lagrangian description of the charged particle motion. In the SOL, due to the large fluctuations and the short characteristic gradient width $L_\phi \sim L_p$, a strong electric field is present. To properly retain the effect of a non-negligible $\mathbf{E} \times \mathbf{B}$ drift, $\mathbf{v}_E = -\nabla\phi \times \mathbf{B}/B^2$ ($\mathbf{E} = -\nabla\phi$ in the electrostatic limit employed here), in the equations of motion, we split the perpendicular component of the particle velocity $\mathbf{v}_\perp$ into $\mathbf{v}_\perp = \mathbf{v}_E + \mathbf{v}'_\perp$. In the particle Lagrangian, we keep the resulting term $m\mathbf{v}_E^2/2$ associated with the $\mathbf{E} \times \mathbf{B}$ motion of the gyrocentres, as it will be shown to be of the same order of magnitude as the first-order terms in the Lagrangian (see Krommes (2013) for a discussion on the physical interpretation of this term).

In the kinetic equation, we expand the gyroaveraged distribution function into a Hermite–Laguerre basis, and express the moments of the collision operator in a series of products of the expansion coefficients of the distribution function. For like-species collisions, this expansion is based on the work of Ji & Held (2009), while we make use of the small mass ratio approximation to obtain electron–ion and ion–electron operators that ensure basic conservation properties. The system is closed by Poisson’s equation, involving explicitly the moments of the distribution function, accurate up to order ϵ^2 (we also present a derivation of Poisson’s equation that rigorously includes collisional ϵ_v effects).

This paper is organized as follows. Section 2 derives the equations of motion of a charged particle in the SOL, and the drift-kinetic Boltzmann equation in a conservative form. In § 3 we expand the gyroaveraged distribution function in a Hermite–Laguerre basis and obtain the guiding-centre moments of the collision operator. In § 4 we take moments of the drift-kinetic Boltzmann equation, and deduce the moment-hierarchy equations. Section 5 presents the guiding-centre Poisson equation, accurate up to order ϵ^2 . Finally, in § 6, a fluid model based on the truncation of the Hermite–Laguerre expansion in the high-collisionality regime is presented. The conclusions follow. Appendix A presents the transformation between pitch-angle and parallel–perpendicular velocity bases. Appendix B lists explicitly the moments of the parallel acceleration phase-space-conserving term. Appendix C derives Poisson’s equation with higher-order collisional effects. Finally, in appendix D, the lower-order guiding-centre moments of the collision operator are given in explicit form.

2. SOL guiding-centre model

2.1. Single particle motion

To derive a convenient equation of motion in the presence of a strong magnetic field $\mathbf{B}$, we start with the Hamiltonian of a charged particle of species a (Jackson 1998),

$$H_a(\mathbf{x}, \mathbf{p}) = \frac{[\mathbf{p} - q_a \mathbf{A}]^2}{2m_a} + q_a \phi, \quad (2.1)$$

and its associated Lagrangian,

$$L_a(\mathbf{x}, \mathbf{v}) = [q_a \mathbf{A}(\mathbf{x}) + m_a \mathbf{v}] \cdot \dot{\mathbf{x}} - \left(\frac{m_a v^2}{2} + q_a \phi(\mathbf{x}) \right), \quad (2.2)$$

where $\mathbf{p} = m_a \mathbf{v} + q_a \mathbf{A}$, $\mathbf{v}$ is the particle velocity, $\mathbf{A}$ is the magnetic vector potential, ϕ is the electrostatic potential, m_a is the mass of the particle and q_a its charge.

We now perform a coordinate transformation from the phase-space coordinates $\mathbf{z} = (\mathbf{x}, \mathbf{v})$ to the guiding-centre coordinates $\mathbf{Z} = (\mathbf{R}, v_{\parallel}, \mu, \theta)$ by writing the particle velocity as (see, e.g. Littlejohn (1983))

$$\mathbf{v} = \mathbf{U} + v'_{\perp} \mathbf{c}, \quad (2.3)$$

with $\mathbf{U} = \mathbf{v}_E(\mathbf{R}) + v_{\parallel} \mathbf{b}(\mathbf{R})$, $v_{\parallel} = \mathbf{v} \cdot \mathbf{b}$, $\mathbf{b} = \mathbf{B}/B$ and $\mathbf{v}_E = \mathbf{E} \times \mathbf{B}/B^2$. The gyroangle $\theta = \tan^{-1}[(\mathbf{v} - \mathbf{U}) \cdot \mathbf{e}_2 / (\mathbf{v} - \mathbf{U}) \cdot \mathbf{e}_1]$ is introduced by defining the right-handed coordinate set

$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{b})$, such that $\mathbf{c} = -\mathbf{a} \times \mathbf{b} = \mathbf{a}'(\theta)$, with $\mathbf{a} = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2$. The decomposition in (2.3) allows us to isolate the high-frequency gyromotion, contained in the $v'_\perp \mathbf{c}$ term, from the dominant guiding-centre velocity $\mathbf{U}$. The adiabatic moment μ is defined as

$$\mu = \frac{m_a v_\perp'^2}{2B}, \quad (2.4)$$

whereas the guiding-centre position is

$$\mathbf{R} = \mathbf{x} - \rho_a \mathbf{a}, \quad (2.5)$$

with $\rho_a = \sqrt{2m_a \mu / (q_a^2 B)}$ the Larmor radius. Incidentally, for the case of weakly varying magnetic fields, equation (2.5) describes the circular motion of a particle around its guiding centre $\mathbf{R}$ with radius ρ_a , i.e. $(\mathbf{x} - \mathbf{R})^2 = \rho_a^2$.

As our goal is to develop a model that describes turbulent fluctuations occurring on a spatial scale longer than the sound Larmor radius ρ_s , and a time scale larger than the gyromotion one, we keep terms in the Lagrangian up to $O(\epsilon)$ and order $T_i \lesssim T_e$, which implies

$$k_\perp \rho_i \lesssim \epsilon. \quad (2.6)$$

We therefore expand the electromagnetic fields around $\mathbf{R}$, to first order in ϵ , i.e.

$$\phi(\mathbf{x}) \simeq \phi(\mathbf{R}) + \rho_a \mathbf{a} \cdot \nabla_{\mathbf{R}} \phi(\mathbf{R}), \quad (2.7)$$

and similarly for $\mathbf{A}$. In the following, if not specified, the electromagnetic fields and potentials are evaluated at the guiding-centre position $\mathbf{R}$, and we denote $\nabla = \nabla_{\mathbf{R}}$. In addition, to take advantage of the difference between the turbulent and gyromotion time scales, we use the gyroaveraged Lagrangian $\langle L_a \rangle$ to evaluate the plasma particle motion, where the gyroaveraging operator $\langle \chi \rangle$ acting on a quantity $\chi(\theta)$ is defined as

$$\langle \chi \rangle = \frac{1}{2\pi} \int_0^{2\pi} \chi(\theta) d\theta, \quad (2.8)$$

which is performed at fixed position $\mathbf{x}$, as opposed to the gyrokinetic equation that can be obtained by gyroaveraging with $\mathbf{R}$ fixed (Hazeltine & Meiss 2003).

To evaluate $\langle L_a \rangle$ we note that, with the expansion for ϕ and $\mathbf{A}$, the Lagrangian in (2.2) can be expressed as $L_a = L_{0a} + L_{1a} + \tilde{L}_a$ where L_{0a} is gyroangle independent,

$$L_{0a} = (q_a \mathbf{A} + m_a \mathbf{U}) \cdot \dot{\mathbf{R}} - \left(\frac{m_a v_\parallel^2}{2} + \frac{m_a v_E^2}{2} + \mu B + q_a \phi \right), \quad (2.9)$$

L_{1a} is proportional to ρ_a^2 (and hence to μ),

$$L_{1a} = \rho_a^2 q_a \dot{\theta} (\mathbf{a} \cdot \nabla) (\mathbf{A} \cdot \mathbf{c}) + m_a \rho_a^2 \Omega \dot{\theta} + \rho_a \dot{\rho}_a [q_a (\mathbf{a} \cdot \nabla) (\mathbf{A} \cdot \mathbf{a})], \quad (2.10)$$

and the $\tilde{L}_a$ contribution contains the terms linearly proportional to $\cos \theta$ or $\sin \theta$ (Cary & Brizard 2009) which are not present in $\langle L_a \rangle$ as $\langle \tilde{L}_a \rangle = 0$.

We note that $\langle L_{1a} \rangle$ can be simplified since $\langle (\mathbf{a} \cdot \nabla) \mathbf{A} \cdot \mathbf{c} \rangle = -\mathbf{b} \cdot (\nabla \times \mathbf{A})/2$, and $\langle (\mathbf{a} \cdot \nabla) \mathbf{A} \cdot \mathbf{a} \rangle = \nabla_\perp \cdot \mathbf{A}/2$. Subtracting the total derivative $-q_a d/dt(\rho_a^2 \nabla_\perp \mathbf{A})/4$ from $\langle L_a \rangle$, which does not alter the resulting equations of motion, we redefine the gyroaveraged Lagrangian as

$$\begin{aligned} \langle L_a \rangle = & (q_a \mathbf{A} + m_a \mathbf{U}) \cdot \dot{\mathbf{R}} - \left(\frac{m_a v_{\parallel}^2}{2} + \frac{m_a v_E^2}{2} + q_a \phi \right) \\ & - \mu B \left(1 - \frac{\dot{\theta}}{\Omega_a} \right) - \frac{\rho_a^2}{4} \frac{d}{dt} [\nabla_{\perp} \cdot (q_a \mathbf{A})]. \end{aligned} \quad (2.11)$$

We now order the terms appearing in $\langle L_a \rangle$. As imposed by the Bohm sheath conditions (Stangeby 2000), both electrons and ions stream along the field lines with parallel velocities comparable to the sound speed $c_s = \sqrt{T_e/m_i}$ in the SOL. The Bohm boundary conditions at the sheath also set the electrostatic potential $e\phi \sim \Lambda T_e$ across the SOL, where $\Lambda = \log \sqrt{m_i/(m_e 2\pi)} \simeq 3$. Therefore, we keep the $m_a v_E^2/2$ term in the Lagrangian in (2.11), as to take into account the presence of the numerically large factor Λ^2 in $v_E^2 \sim \epsilon^2 \Lambda^2 c_s^2$.

By neglecting the higher-order terms in (2.11), i.e. $-(\rho_a^2/4)d[\nabla_{\perp} \cdot (q_a \mathbf{A})]/dt$, the expression for the gyroaveraged Lagrangian describing SOL single particle dynamics, up to $O(\epsilon)$, can be written as

$$\langle L_a \rangle = q_a \mathbf{A}^* \cdot \dot{\mathbf{R}} - q_a \phi^* - \frac{m_a v_{\parallel}^2}{2} + \mu \frac{m_a \dot{\theta}}{q_a}, \quad (2.12)$$

where $q_a \phi^* = q_a \phi + m_a v_E^2/2 + \mu B$, and $q_a \mathbf{A}^* = q_a \mathbf{A} + m_a v_{\parallel} \mathbf{b} + m_a \mathbf{v}_E$. The Euler-Lagrange equations applied to the Lagrangian in (2.12) for the coordinates θ , $v_{\parallel}$, and μ , yield, respectively, $\dot{\mu} = 0$, $v_{\parallel} = \mathbf{b} \cdot \dot{\mathbf{R}}$, and $\dot{\theta} = \Omega_a$. For the $\mathbf{R}$ coordinate, we obtain

$$m_a \dot{v}_{\parallel} \mathbf{b} = q_a (\mathbf{E}^* + \dot{\mathbf{R}} \times \mathbf{B}^*), \quad (2.13)$$

where the relation $(\nabla \mathbf{A} - (\nabla \mathbf{A})^T) \cdot \dot{\mathbf{R}} = \dot{\mathbf{R}} \times (\nabla \times \mathbf{A})$ has been used, and we defined $\mathbf{E}^* = -\nabla \phi^* - \partial_t \mathbf{A}^*$, and $\mathbf{B}^* = \nabla \times \mathbf{A}^*$, with the parallel component of $\mathbf{B}^*$ given by

$$B_{\parallel}^* = \mathbf{B}^* \cdot \mathbf{b} = B + \frac{m_a}{q_a} \mathbf{b} \cdot \nabla \times (v_{\parallel} \mathbf{b} + \mathbf{v}_E). \quad (2.14)$$

By projecting (2.13) along $\mathbf{B}^*$, we derive $m \dot{v}_{\parallel} B_{\parallel}^* = e \mathbf{E}^* \cdot \mathbf{B}^*$, while crossing with $\mathbf{b}$ yields the guiding-centre velocity $\dot{\mathbf{R}} B_{\parallel}^* = v_{\parallel} \mathbf{B}^* + \mathbf{E}^* \times \mathbf{B}/B$. Using the expressions for the fields $\mathbf{E}^*$ and $\mathbf{B}^*$, we obtain

$$\dot{\mathbf{R}} = \mathbf{U} + \frac{\mathbf{B}}{\Omega_a B_{\parallel}^*} \times \left(\frac{d\mathbf{U}}{dt} + \frac{\mu \nabla B}{m_a} \right), \quad (2.15)$$

and

$$m_a \dot{v}_{\parallel} = q_a E_{\parallel} - \mu \nabla_{\parallel} B + m_a \mathbf{v}_E \cdot \frac{d\mathbf{b}}{dt} - m_a \mathcal{A}. \quad (2.16)$$

In (2.15), (2.16), in addition to the time derivatives of the phase-space coordinates $\dot{\mathbf{R}}$, $\dot{v}_{\parallel}$, that only have an explicit time dependence, we define the total derivative d/dt of a field $\phi(\mathbf{R}, t)$ that has an explicit time and $\mathbf{R}$ dependence as $d\phi/dt \equiv \partial_t \phi + \mathbf{U} \cdot \nabla \phi$. The $\mathcal{A}$ term represents the higher-order nonlinear terms in $\dot{v}_{\parallel}$ that ensure phase-space conservation properties (Cary & Brizard 2009), and it is given by

$$\mathcal{A} = \frac{B}{B_{\parallel}^*} \left(\frac{d\mathbf{U}}{dt} \Big|_{\perp} + \mu \nabla_{\perp} B \right) \cdot \frac{\nabla \times \mathbf{U}}{\Omega_a}. \quad (2.17)$$

The guiding-centre equations of motion (2.15) and (2.16) satisfy the energy, $E_{gc} = q_a \phi^* + m_a v_{\parallel}^2/2$ (Cary & Brizard 2009), and momentum, $\mathbf{P}_{gc} = e\mathbf{A}^*$ (Cary & Brizard 2009), conservation laws, i.e.

$$\frac{dE_{gc}}{dt} = q_a \frac{\partial \phi^*}{\partial t} - q_a \frac{\partial \mathbf{A}^*}{\partial t} \cdot \dot{\mathbf{R}}, \quad (2.18)$$

and

$$\frac{\partial \mathbf{P}_{gc}}{\partial t} = -q_a \nabla \phi^* + q_a \nabla \mathbf{A}^* \cdot \dot{\mathbf{R}}. \quad (2.19)$$

In addition, we note that using (2.15), (2.16) and Maxwell's equations, a conservation equation for $B_{\parallel}^*$ can be derived:

$$\frac{\partial B_{\parallel}^*}{\partial t} + \nabla \cdot (\dot{\mathbf{R}} B_{\parallel}^*) + \frac{\partial}{\partial v_{\parallel}} (\dot{v}_{\parallel} B_{\parallel}^*) = 0. \quad (2.20)$$

2.2. The guiding-centre Boltzmann equation

The Boltzmann equation for the evolution of the distribution function $f_a(\mathbf{x}, \mathbf{v})$ of the particles in $(\mathbf{x}, \mathbf{v})$ coordinates is

$$\frac{\partial f_a}{\partial t} + \dot{\mathbf{x}} \cdot \nabla_{\mathbf{x}} f_a + \dot{\mathbf{v}} \cdot \nabla_{\mathbf{v}} f_a = C(f_a), \quad (2.21)$$

where $C(f_a) = \sum_b C(f_a, f_b) = \sum_b C_{ab}$ is the collision operator. Because f_a can significantly deviate from a Maxwellian distribution function in the SOL (Battaglia *et al.* 2014), we consider the bilinear Coulomb operator C_{ab} (Balescu 1988), to model collisions between particles of species a and b

$$C_{ab} = L_{ab} \frac{\partial}{\partial v_i} \left[\frac{\partial^2 G_b}{\partial v_i \partial v_j} \frac{\partial f_a}{\partial v_j} - \frac{m_a}{m_b} \frac{\partial H_b}{\partial v_i} f_a \right], \quad (2.22)$$

with

$$H_b = 2 \int \frac{f_b(\mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|} d\mathbf{v}', \quad (2.23)$$

and

$$G_b = \int f_b(\mathbf{v}') |\mathbf{v} - \mathbf{v}'| d\mathbf{v}', \quad (2.24)$$

the Rosenbluth potentials satisfying $\nabla_{\mathbf{v}}^2 G_b = H_b$. In (2.22) we introduced $L_{ab} = q_a^2 q_b^2 \lambda / (4\pi \epsilon_0^2 m_a^2) = \nu_{ab} v_{tha}^3 / n_b$, where λ is the Coulomb logarithm, ν_{ab} the collision frequency between species a and b and $v_{tha}^2 = 2T_a/m_a$.

Taking advantage of the small electron to ion mass ratio, the collision operator between unlike species can be simplified (see, e.g. Balescu (1988), Helander & Sigmar (2005)). The electron-ion collisions are modelled by using the operator $C_{ei}(f_e) = C_{ei}^0 + C_{ei}^1$, where C_{ei}^0 is the Lorentz pitch-angle scattering operator

$$C_{ei}^0 = \frac{n_i L_{ei}}{8\pi} \frac{\partial}{\partial c_e} \cdot \left[\frac{1}{c_e} \frac{\partial f_e}{\partial c_e} - \frac{\mathbf{c}_e}{c_e^3} \left(\mathbf{c}_e \cdot \frac{\partial f_e}{\partial \mathbf{c}_e} \right) \right], \quad (2.25)$$

and C_{ei}^1 the momentum-conserving term

$$C_{ei}^1 = \frac{n_i L_{ei}}{4\pi c_e^3} f_{Me} \mathbf{u}_i \cdot \mathbf{c}_e. \quad (2.26)$$

with $\mathbf{c}_e = (\mathbf{v} - \mathbf{u}_e)/v_{the}$ the normalized difference between the electron particle $\mathbf{v}$ and fluid $\mathbf{u}_e$ velocities, and $\mathbf{u}_i$ the ion fluid velocity.

Ion–electron collisions are modelled with the operator

$$C_{ie} = \frac{\mathbf{R}_{ei}}{m_i n_i v_{thi}} \cdot \frac{\partial f_i}{\partial \mathbf{c}_i} + v_{ei} \frac{n_e m_e}{n_i m_i} \frac{\partial}{\partial \mathbf{c}_i} \cdot \left(\mathbf{c}_i f_i + \frac{T_e}{T_i} \frac{\partial f_i}{\partial \mathbf{c}_i} \right), \quad (2.27)$$

where $\mathbf{R}_{ei} = \int m_e \mathbf{v} C_{ei} d\mathbf{v}$ is the electron–ion friction force.

We take advantage of (1.2) to order the electron collision frequency ν_e and the ion collision frequency ν_i as

$$\frac{\nu_i}{\Omega_i} \sim \sqrt{\frac{m_e}{m_i}} \left(\frac{T_e}{T_i} \right)^{3/2} \epsilon_v \lesssim \epsilon^2, \quad (2.28)$$

where we used the relation $\nu_i \sim \sqrt{m_e/m_i} (T_e/T_i)^{3/2} \nu_e$. The orderings in (2.7) and (2.28) yield the lower bound in (1.3) for the ion to electron temperature ratio. We now express the particle distribution function f_a in terms of the guiding-centre coordinates by defining F_a , a function of guiding-centre coordinates, as

$$F_a(\mathbf{R}, v_{\parallel}, \mu, \theta) = f_a(\mathbf{x}(\mathbf{R}, v_{\parallel}, \mu, \theta), \mathbf{v}(\mathbf{R}, v_{\parallel}, \mu, \theta)). \quad (2.29)$$

Using the chain rule to rewrite (2.21) in guiding-centre coordinates, we obtain

$$\frac{\partial F_a}{\partial t} + \dot{\mathbf{R}} \cdot \nabla F_a + \dot{v}_{\parallel} \frac{\partial F_a}{\partial v_{\parallel}} + \dot{\mu} \frac{\partial F_a}{\partial \mu} + \dot{\theta} \frac{\partial F_a}{\partial \theta} = C(F_a), \quad (2.30)$$

where $\dot{\mathbf{R}}$ and $\dot{v}_{\parallel}$ are given by (2.15) and (2.16) respectively, $\dot{\theta} = \Omega_a$, and $\dot{\mu} = 0$. Equation (2.30) can be simplified by applying the gyroaveraging operator. This results in the drift-kinetic equation

$$\frac{\partial \langle F_a \rangle}{\partial t} + \dot{\mathbf{R}} \cdot \nabla \langle F_a \rangle + \dot{v}_{\parallel} \frac{\partial \langle F_a \rangle}{\partial v_{\parallel}} = \langle C(F_a) \rangle. \quad (2.31)$$

We now write (2.31) in a form useful to take gyrofluid moments of the form $\int \langle F_a \rangle B dv_{\parallel} d\mu d\theta$ (see §4). Using the conservation law in (2.20) for $B_{\parallel}^*$, we can write the guiding-centre Boltzmann equation in conservative form as

$$\frac{\partial (B_{\parallel}^* \langle F_a \rangle)}{\partial t} + \nabla \cdot (\dot{\mathbf{R}} B_{\parallel}^* \langle F_a \rangle) + \frac{\partial (\dot{v}_{\parallel} B_{\parallel}^* \langle F_a \rangle)}{\partial v_{\parallel}} = B_{\parallel}^* \langle C(F_a) \rangle. \quad (2.32)$$

Moreover, in order to relate the gyrofluid moments $\int \langle F_a \rangle B dv_{\parallel} d\mu d\theta$ with the usual fluid moments $\int f_a d^3v$, we estimate the order of magnitude of the gyrophase dependent part of the distribution function $\tilde{F}_a = F_a - \langle F_a \rangle$ where $\langle F_a \rangle$ obeys (2.31).

The equation for the evolution of $\tilde{F}_a$ is obtained by subtracting (2.31) from the Boltzmann equation, (2.30), that is

$$\frac{\partial \tilde{F}_a}{\partial t} + \dot{\mathbf{R}} \cdot \nabla \tilde{F}_a + \dot{v}_{\parallel} \frac{\partial \tilde{F}_a}{\partial v_{\parallel}} + \Omega_a \frac{\partial \tilde{F}_a}{\partial \theta} = C(F_a) - \langle C(F_a) \rangle. \quad (2.33)$$

Using the orderings in (1.2) and (2.28), as well as $\partial_t \sim \dot{\mathbf{R}} \cdot \nabla \sim \dot{v}_{\parallel} \partial_{v_{\parallel}} \sim \epsilon \Omega_i$ and $\Omega_a \partial_{\theta} \sim \Omega_a$, the comparison of the leading-order term on the left-hand side of (2.33) with the right-hand side of the same equation imply the following ordering for $\tilde{F}_e$

$$\frac{\tilde{F}_e}{\langle F_e \rangle} \sim \frac{m_e}{m_i} \epsilon_v \lesssim \epsilon^2, \quad (2.34)$$

and $\tilde{F}_i$

$$\frac{\tilde{F}_i}{\langle F_i \rangle} \sim \sqrt{\frac{m_e}{m_i}} \left(\frac{T_e}{T_i} \right)^{3/2} \epsilon_v \lesssim \epsilon^2. \quad (2.35)$$

To evaluate the leading-order term of $\tilde{F}_a$, we expand the collision operator $C(F_a) = C_0(\langle F_a \rangle) + \epsilon C_1(F_a) + \dots$, such that

$$\tilde{F}_a \simeq \frac{1}{\Omega_a} \int_0^{\theta} [C_0(\langle F_a \rangle) - \langle C_0(\langle F_a \rangle) \rangle] d\theta' + O(\epsilon^3 \langle F_a \rangle). \quad (2.36)$$

The relation in (2.36) can be further simplified by expanding the θ dependence of F_a in Fourier harmonics,

$$F_a = \sum_m e^{im\theta} F_{ma}, \quad (2.37)$$

so that for $m=0$ we have $\langle F_a \rangle = F_{0a}$, and similarly for $C_0(\langle F_a \rangle)$

$$C_0(\langle F_a \rangle) = \sum_{m'} e^{im'\theta} C_{m'a}. \quad (2.38)$$

We can then write (2.36) as

$$\tilde{F}_{ma} = \frac{C_{ma}}{im\Omega_a}, \quad (2.39)$$

for $m \neq 0$.

3. Moment expansion

We now derive a polynomial expansion for the distribution function $\langle F_a \rangle$ that simplifies the solution of (2.32), with the collision operators in (2.22)–(2.27). This section is organized as follows. In §3.1 the Hermite–Laguerre basis is introduced, relating the corresponding expansion coefficients for $\langle F_a \rangle$ with its usual gyrofluid moments. In §3.2, we briefly review the fluid moment expansion of the Coulomb collision operator presented in Ji & Held (2006, 2008). In §3.3, leveraging the work in Ji & Held (2006, 2008), we expand C_{ab} in terms of the product of the gyrofluid moments, for both like- and unlike-species collisions. This ultimately gives us the possibility of solving (2.32) in terms of gyrofluid moments.

3.1. Guiding-centre moment expansion of $\langle F_a \rangle$

To take advantage of the anisotropy introduced by a strong magnetic field, and efficiently treat the left-hand side of (2.32) where the parallel and perpendicular directions appear decoupled, we express $\langle F_a \rangle$ by using a Hermite polynomial basis expansion for the parallel velocity coordinate (Grad 1949; Armstrong 1967; Grant & Feix 1967; Ng, Bhattacharjee & Skiff 1999; Zocco & Schekochihin 2011; Loureiro, Schekochihin & Zocco 2013; Parker & Dellar 2015; Schekochihin *et al.* 2016; Tassi 2016) and a Laguerre polynomial basis for the perpendicular velocity coordinate (Omotani *et al.* 2015; Zocco *et al.* 2015; Mandell *et al.* 2017). More precisely, we use the following expansion

$$\langle F_a \rangle = \sum_{p,j=0}^{\infty} \frac{N_a^{pj}}{\sqrt{2^p p!}} F_{Ma} H_p(s_{\parallel a}) L_j(s_{\perp a}^2), \quad (3.1)$$

where the physicists' Hermite polynomials H_p of order p are defined by Abramowicz *et al.* (1988)

$$H_p(x) = (-1)^p e^{x^2} \frac{d^p}{dx^p} e^{-x^2}, \quad (3.2)$$

and normalized via

$$\int_{-\infty}^{\infty} dx H_p(x) H_{p'}(x) e^{-x^2} = 2^p p! \sqrt{\pi} \delta_{pp'}, \quad (3.3)$$

and the Laguerre polynomials L_j of order j are defined by Abramowicz *et al.* (1988)

$$L_j(x) = \frac{e^x}{j!} \frac{d^j}{dx^j} (e^{-x} x^j), \quad (3.4)$$

which are orthonormal with respect to the weight e^{-x}

$$\int_0^{\infty} dx L_j(x) L_{j'}(x) e^{-x} = \delta_{jj'}. \quad (3.5)$$

Because of the orthogonality of the Hermite–Laguerre basis, the coefficients N_a^{pj} of the expansion in (3.1) are

$$N_a^{pj} = \frac{1}{N_a} \int \frac{H_p(s_{\parallel a}) L_j(s_{\perp a}^2) \langle F_a \rangle}{\sqrt{2^p p!}} \frac{B}{m_a} d\mu dv_{\parallel} d\theta, \quad (3.6)$$

and correspond to the guiding-centre moments of $\langle F_a \rangle$.

In (3.1), the shifted bi-Maxwellian is introduced

$$F_{Ma} = N_a \frac{e^{-s_{\parallel a}^2 - s_{\perp a}^2}}{\pi^{3/2} v_{th\parallel a} v_{th\perp a}^2}, \quad (3.7)$$

where $s_{\parallel a}$ and $s_{\perp a}$ are the normalized parallel and perpendicular shifted velocities respectively, defined by

$$s_{\parallel a} = \frac{v_{\parallel} - u_{\parallel a}}{v_{th\parallel a}}, \quad v_{th\parallel a}^2 = \frac{2T_{\parallel a}}{m_a}, \quad (3.8a,b)$$

and

$$s_{\perp a}^2 = \frac{v_{\perp}^2}{v_{th\perp a}^2} = \frac{\mu B}{T_{\perp a}}, \quad v_{th\perp a}^2 = \frac{2T_{\perp a}}{m_a}, \quad (3.9a,b)$$

which provide an efficient representation of the distribution function in both the weak ($u_{\parallel a} \ll v_{tha}$) and strong flow ($u_{\parallel a} \sim v_{tha}$) regimes.

The guiding-centre density N_a , appearing in (3.7), the guiding-centre fluid velocity $u_{\parallel a}$, in (3.8) and the guiding-centre parallel $T_{\parallel a} = P_{\parallel a}/N_a$ and perpendicular $T_{\perp a} = P_{\perp a}/N_a$ temperatures in (3.8), (3.9) are defined as $N_a = \|1\|_a$, $N_a u_{\parallel a} = \|v_{\parallel}\|_a$, $P_{\parallel a} = m_a \|(v_{\parallel} - u_{\parallel a})^2\|_a$ and $P_{\perp a} = \|\mu B\|_a$, where

$$\|\chi\|_a \equiv \int \chi \langle F_a \rangle \frac{B}{m_a} d\mu dv_{\parallel} d\theta. \quad (3.10)$$

The definitions of N_a , $u_{\parallel a}$, $P_{\parallel a}$ and $P_{\perp a}$ imply that $N_a^{00} = 1$, $N_a^{10} = 0$, $N_a^{20} = 0$, $N_a^{01} = 0$ respectively. Later, we will consider the parallel and perpendicular heat flux, defined as

$$Q_{\parallel a} = m_a \|(v_{\parallel} - u_{\parallel a})^3\|_a, \quad Q_{\perp a} = \|(v_{\parallel} - u_{\parallel a})\mu B\|_a, \quad (3.11a,b)$$

which are related to the coefficients N_a^{30} , N_a^{11} by

$$N_a^{30} = \frac{Q_{\parallel a}}{\sqrt{3}P_{\parallel a}v_{tha\parallel}}, \quad N_a^{11} = -\frac{\sqrt{2}Q_{\perp a}}{P_{\perp a}v_{tha\parallel}}. \quad (3.12a,b)$$

3.2. Fluid moment expansion of the collision operator

A polynomial expansion of the collision operators in (2.22) was carried out in Ji & Held (2006), and later extended to effectively take into account finite fluid velocity and unlike-species collisions in Ji & Held (2008). This allowed expressing C_{ab} as products of fluid moments of f_a and f_b . We summarize here the main steps of Ji & Held (2006, 2008).

Similarly to (3.1), the particle distribution function f_a is expanded as

$$f_a = f_{aM} \sum_{l,k=0}^{\infty} \frac{L_k^{l+1/2}(c_a^2) \mathbf{P}^l(\mathbf{c}_a) \cdot \mathbf{M}_a^{lk}}{\sqrt{\sigma_k^l}}, \quad (3.13)$$

where $f_{aM} = n_a \exp(-c_a^2)/(\pi^{3/2}v_{tha}^3)$ is a shifted Maxwell–Boltzmann distribution function, and $\mathbf{c}_a$ the shifted velocity defined as $\mathbf{c}_a = (\mathbf{v} - \mathbf{U}_a)/v_{tha}$, with $\mathbf{U}_a$ the fluid velocity. The fluid variables n_a , $\mathbf{U}_a$, and T_a are defined as the usual moments of the particle distribution function f_a , i.e. $n_a = \int f_a d^3\mathbf{v}$, $n_a \mathbf{U}_a = \int f_a \mathbf{v} d^3\mathbf{v}$, $n_a T_a = \int m f_a (\mathbf{v} - \mathbf{U}_a)^2 d^3\mathbf{v}/3$.

The tensors $\mathbf{P}_a^{lk}(\mathbf{c}_a) = \mathbf{P}^l(\mathbf{c}_a) L_k^{l+1/2}(c_a^2)$ constitute an orthogonal basis, where $\mathbf{P}^l(\mathbf{c}_a)$ is the symmetric and traceless tensor

$$\mathbf{P}^l(\mathbf{c}_a) = \sum_{i=0}^{\lfloor l/2 \rfloor} d_i^l S_i^l c_a^{2i} \{\mathbf{I}^l \mathbf{c}_a^{l-2i}\}, \quad (3.14)$$

with $\mathbf{I}$ denoting the identity matrix, $\{\mathbf{A}^i\}$ denoting the symmetrization of the tensor $\mathbf{A}^i$, $[l/2]$ denoting the largest integer less than or equal to $l/2$, and the coefficients d_i^l and S_i^l defined by

$$d_i^l = \frac{(-2)^i (2l - 2i)! l!}{(2l)! (l - i)!}, \quad (3.15)$$

and

$$S_i^l = \frac{l!}{(l - 2i)! 2^i i!}. \quad (3.16)$$

The tensor $\mathbf{P}^l(\mathbf{c}_a)$ is normalized via

$$\int d\mathbf{v} \mathbf{P}^n(\mathbf{v}) \mathbf{P}^l(\mathbf{v}) \cdot \mathbf{M}^l g(\mathbf{v}) = \mathbf{M}^n \delta_{n,l} \sigma_n \int d\mathbf{v} v^{2n} g(\mathbf{v}), \quad (3.17)$$

with $\sigma_l = l!/[2^l(l+1/2)!]$. We note that the tensor $\mathbf{A}^i$ is formed by i multiplications of the $\mathbf{A}$ elements (e.g. if $\mathbf{A}$ is a rank-2 tensor, $\mathbf{A}^3 \equiv \mathbf{A}\mathbf{A}\mathbf{A}$, which in index notation can be written as $(A^3)_{ijklmn} = A_{ij}A_{lk}A_{mn}$).

In the expansion in (3.13), $L_k^{l+1/2}(x)$ are the associated Laguerre polynomials

$$L_k^{l+1/2}(x) = \sum_{m=0}^k L_{km}^l x^m, \quad (3.18)$$

normalized via

$$\int_0^\infty e^{-x} x^{l+1/2} L_k^{l+1/2}(x) L_{k'}^{l+1/2}(x) dx = \lambda_k^l \delta_{k,k'}, \quad (3.19)$$

with $\lambda_k^l = (l+k+1/2)!/k!$ and $L_{km}^l = [(-1)^m (l+k+1/2)!]/[(k-m)!(l+m+1/2)!m!]$. The $\sigma_k^l = \sigma_l \lambda_k^l$ term is a normalization factor from the orthogonality relations in (3.17), (3.19).

Finally, the coefficients of the expansion in (3.13) $\mathbf{M}_a^{lk}$ are

$$\mathbf{M}_a^{lk} = \frac{1}{n_a} \int d\mathbf{v} f_a \frac{L_k^{l+1/2}(c_a^2) \mathbf{P}^l(\mathbf{c}_a)}{\sqrt{\sigma_k^l}}, \quad (3.20)$$

which correspond to the moments of f_a due to the orthogonality relations in (3.17), (3.19).

By using the expansion in (3.13) in the collision operator in (2.22), a closed form for C_{ab} in terms of products of $\mathbf{M}_a^{lk}$ can be obtained. For like-species collisions it reads

$$C_{aa} = \sum_{lkm} \sum_{nqr} \frac{L_{km}^l L_{qr}^n}{\sqrt{\sigma_k^l \sigma_q^n}} c(f_a^{lkm}, f_a^{nqr}), \quad (3.21)$$

with

$$c(f_a^{lkm}, f_a^{nqr}) = f_a M \sum_{u=0}^{\min(2,l,n)} v_{*aa}^{lm,nr}(c_a^2) \sum_{i=0}^{\min(l,n)-u} d_i^{l-u,n-u} \mathbf{P}^{l+n-2(i+u)}(\hat{\mathbf{c}}_a) \cdot \overline{\mathbf{M}_a^{lk} \cdot \mathbf{M}_a^{nq}}, \quad (3.22)$$

where $\hat{\mathbf{c}}_a = \mathbf{c}_a/c_a$, $\cdot^n$ is the n -fold inner product (e.g. for the matrix $\mathbf{A} = A_{ij}$, $(\mathbf{A} \cdot^1 \mathbf{A})_{ij} = \sum_k A_{ki} A_{kj}$) and $\overline{\mathbf{A}}$ the traceless symmetrization of $\mathbf{A}$ (e.g. $\overline{\mathbf{A}} = (A_{ij} + A_{ji})/2 - \delta_{ij} \sum_k A_{kk}/3$). We refer the reader to Ji & Held (2009) for the explicit form of the $v_{*abu}^{lm,nr}$ coefficients. The expansion of the unlike-species collisions is reported in Ji & Held (2008).

3.3. Guiding-centre moment expansion of the collision operators

In order to apply the gyroaveraging operator to the like-species collision operator C_{aa} in (3.21), we expand the fluid moments as $\mathbf{M}_a^{lk} = \mathbf{M}_{a0}^{lk} + \epsilon \mathbf{M}_{a1}^{lk} + \dots$, aiming at representing the collision operator up to $O(\epsilon_v \epsilon)$. An analytical expression for the leading order $\mathbf{M}_{a0}^{lk}$ in terms of guiding-centre moments N_a^{pj} can be obtained as follows. By splitting $f_a = \langle f_a \rangle + \tilde{f}_a$ when evaluating the fluid moments $\mathbf{M}_a^{lk}$ according to (3.20), we obtain

$$\mathbf{M}_a^{lk} = \frac{1}{n_a} \int d^3x' d^3v' \delta(\mathbf{x}' - \mathbf{x}) \frac{L_k^{l+1/2} (c_a'^2) \mathbf{P}^l(\mathbf{c}_a')}{\sqrt{\sigma_k^l}} (\langle f_a \rangle + \tilde{f}_a), \quad (3.23)$$

where the Dirac delta function was introduced to convert the velocity integral into an $(\mathbf{x}, \mathbf{v})$ integral that encompasses the full phase space. Since the volume element in phase space can be written as $d^3\mathbf{x} d^3\mathbf{v} = (B_{\parallel}^*/m) d\mathbf{R} dv_{\parallel} d\mu d\theta$ (Cary & Brizard 2009), and defining $\mathbf{x}' = \mathbf{R} + \rho_a \mathbf{a}$, we can write the fluid moments in (3.23) as

$$\mathbf{M}_a^{lk} = \frac{1}{n_a} \int d\mathbf{R} dv_{\parallel} d\mu d\theta \frac{B_{\parallel}^*}{m_a} \delta(\mathbf{x} - \mathbf{R} - \rho_a \mathbf{a}) \frac{L_k^{l+1/2} (c_a'^2) \mathbf{P}^l(\mathbf{c}_a')}{\sqrt{\sigma_k^l}} (\langle F_a \rangle + \tilde{F}_a), \quad (3.24)$$

where $\langle f_a \rangle$ and $\tilde{f}_a$ in (3.23) are written in terms of guiding-centre coordinates using (2.29). Neglecting the higher order ρ_a and $\tilde{F}_a$ terms, the leading-order fluid moments $\mathbf{M}_{a0}^{lk}$ are given by

$$\mathbf{M}_{a0}^{lk} = \frac{1}{n_a} \int dv_{\parallel} d\mu d\theta \frac{B_{\parallel}^*}{m_a} \frac{L_k^{l+1/2} (c_a'^2) \mathbf{P}^l(\mathbf{c}_a')}{\sqrt{\sigma_k^l}} \langle F_a \rangle. \quad (3.25)$$

The θ integration can be performed by making use of the gyroaveraging formula of the $\mathbf{P}^l$ tensor (Ji & Held 2009)

$$\langle \mathbf{P}^l(\mathbf{c}_a) \rangle = c_a^l P_l(\xi_a) \mathbf{P}^l(\mathbf{b}), \quad (3.26)$$

where $\xi_a = \mathbf{c}_a \cdot \mathbf{b} / c_a$ is the pitch-angle velocity coordinate, and P_l is a Legendre polynomial defined by

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} [(x^2 - 1)^l], \quad (3.27)$$

and normalized via

$$\int_{-1}^1 P_l(x) P_{l'}(x) dx = \frac{\delta_{ll'}}{l + 1/2}, \quad (3.28)$$

yielding

$$\mathbf{M}_{a0}^{lk} = \frac{\mathbf{P}^l(\mathbf{b})}{n_a} \int dv_{\parallel} d\mu d\theta \frac{B_{\parallel}^*}{m_a} \frac{L_k^{l+1/2} (c_a'^2) c_a^l P_l(\xi_a)}{\sqrt{\sigma_k^l}} \langle F_a \rangle. \quad (3.29)$$

Finally, we use the basis transformation

$$c_a^l P_l(\xi_a) L_k^{l+1/2} (c_a'^2) = \sum_{p=0}^{l+2k} \sum_{j=0}^{k+[l/2]} T_{alk}^{pj} H_p(s_{\parallel a}) L_j(s_{\perp a}^2), \quad (3.30)$$

with the inverse

$$H_p(s_{\parallel a})L_j(s_{\perp a}^2) = \sum_{l=0}^{p+2j} \sum_{k=0}^{j+\lfloor p/2 \rfloor} (T_a^{-1})_{pj}^{lk} c_a^l P_l(\xi_a) L_k^{l+1/2}(c_a^2), \quad (3.31)$$

to obtain an expression in terms of the Hermite–Laguerre basis. A numerical evaluation of T_{alk}^{pj} and $(T_a^{-1})_{pj}^{lk}$ was carried out in Omotani *et al.* (2015). Instead, in appendix A, we derive the analytic expressions of both T_{alk}^{pj} and $(T_a^{-1})_{pj}^{lk}$.

Using the definition of guiding-centre moments N_a^{pj} in (3.6), the leading-order fluid moment $\mathbf{M}_{a0}^{lk}$ is then given by

$$n_a \mathbf{M}_{a0}^{lk} = N_a \mathbf{P}^l(\mathbf{b}) \mathcal{N}_a^{lk}, \quad (3.32)$$

where we define

$$\mathcal{N}_a^{lk} = \sum_{p=0}^{l+2k} \sum_{j=0}^{k+\lfloor l/2 \rfloor} T_{alk}^{pj} N_a^{pj} \sqrt{\frac{2^p p!}{\sigma_k^l}}. \quad (3.33)$$

The leading-order part C_{aa0} of the collision operator C_{aa} can be calculated by approximating $\mathbf{M}_a^{lk}$ appearing in (3.22) with $\mathbf{M}_{a0}^{lk}$. For the ions, the largest contribution to $\mathbf{M}_i^{lk} - \mathbf{M}_{i0}^{lk}$ is of order ϵ and it is given by the ρ_i appearing in (3.24) (the $\tilde{F}_i$ correction is smaller since $\tilde{F}_i \lesssim \epsilon^2 \langle F_i \rangle$, see (2.35)). Therefore, by using the ordering in (2.28), the largest correction to C_{ii0} is $O(\sqrt{m_e/m_i} \epsilon \epsilon_v)$. The correction to C_{ee0} is of the same order. It follows that we can approximate C_{aa} appearing in (3.22) with C_{aa0} to represent the collision operator up to $O(\epsilon_v \epsilon)$.

As an aside, we note that the relationship between the guiding-centre and fluid moments in (3.32) provides, for the indices $(l, k) = (0, 0)$,

$$n_a = N_a, \quad (3.34)$$

while, for $(l, k) = (0, 1)$,

$$T_a = \frac{T_{\parallel a} + 2T_{\perp a}}{3}. \quad (3.35)$$

Moreover, the $(l, k) = (2, 0)$ moment provides a relationship useful to express the viscosity tensor $\boldsymbol{\Pi}_a = \int (\mathbf{c}_a \mathbf{c}_a - c_a^2 \mathbf{I}) f_a \, d\mathbf{v}$

$$\boldsymbol{\Pi}_a = \mathbf{b} \mathbf{b} N (T_{\parallel a} - T_{\perp a}), \quad (3.36)$$

and $(l, k) = (1, 1)$ gives

$$\mathbf{q}_a = \left(\frac{Q_{\parallel a}}{2} + Q_{\perp a} \right) \mathbf{b}, \quad (3.37)$$

with $\mathbf{q}_a$ the heat flux density $\mathbf{q}_a = m \int \mathbf{c}_a c_a^2 f_a \, d\mathbf{v} / 2$.

In order to express the Boltzmann equation, (2.32), in terms of the guiding-centre moments N_a^{pj} , we evaluate the guiding-centre moments of $\langle C_{aa} \rangle$, namely

$$C_{aa}^{pj} = \frac{1}{N_a} \int \langle C_{aa0} \rangle \frac{H_p(s_{\parallel a}) L_j(s_{\perp a}^2)}{\sqrt{2^p p!}} \frac{B}{m_a} \, d\mathbf{v}_{\parallel} \, d\mu \, d\theta. \quad (3.38)$$

By using the gyroaveraging property (3.26) of $\mathbf{P}^l(\mathbf{c}_a)$ in the like-species operator in (3.21), (3.22) (with $\mathbf{M}_a^{lk} = \mathbf{M}_{a0}^{lk}$), and the relation (3.32) between $\mathbf{M}_{a0}^{lk}$ and N_a^{pj} , the gyroaveraged collision operator coefficients $\langle c(f_a^{lkm}, f_a^{nqr}) \rangle$ are given by

$$\langle c(f_a^{lkm}, f_a^{nqr}) \rangle = f_{aM} \sum_{u=0}^{\min(2,l,n)} v_{*aau}^{lm,nr} (c_a^2) \sum_{i=0}^{\min(l,n)-u} d_i^{l-u,n-u} P_{l+n-2(i+u)}(\xi) \mathcal{N}_a^{lk} \mathcal{N}_a^{nq} \mathcal{P}_{i+u}^{l,n}, \quad (3.39)$$

with $\mathcal{P}_{i+u}^{l,n} = \mathbf{P}^{l+n-2(i+u)} \cdot \overline{\mathbf{P}^l \cdot \mathbf{i} + u} \mathbf{P}^n$.

Using the basis transformation of (3.31) to express $H_p(s_{\parallel a}) L_j(s_{\perp a}^2)$ in (3.38) in terms of $c_a^l P_l(\xi_a) L_k^{l+1/2}(c_a^2)$, and performing the resulting integral, we obtain

$$C_{aa}^{pj} = \sum_{l,k} \sum_{n,q} \sum_{u=0}^{\min(2,l,n)} \sum_{i=0}^{\min(l,n)-u} \sum_{e=0}^{p+2j} \sum_{f=0}^{j+[p/2]} \sum_{g=0}^f \sum_{m=0}^k \sum_{r=0}^q \times \frac{L_{km}^l L_{qr}^n L_{fg}^e d_i^{l-u,n-u}}{\sqrt{\sigma_k^l \sigma_q^n} (e+1/2) 4\pi} \frac{C_{*aau}^{eg,lm,nr}}{\sqrt{2^p p!}} \delta_{e,l+n-2(i+u)} (T^{-1})_{pj}^{ef} \mathcal{N}_a^{lk} \mathcal{N}_a^{nq} \mathcal{P}_{i+u}^{l,n}, \quad (3.40)$$

with $C_{*aau}^{jw,lm,nr} = \int d\mathbf{v} c_a^{2w+jf} f_{Ma} v_{*aau}^{lm,nr}$.

We now turn to the electron-ion collision operator, $C_{ei} = C_{ei}^0 + C_{ei}^1$, with C_{ei}^0 given by (2.25) and C_{ei}^1 given by (2.26). As the basis $L_k^{l+1/2} \mathbf{P}^l(\mathbf{c}_a)$ is an eigenfunction of the Lorentz pitch-angle scattering operator C_{ei}^0 with eigenvalue $-l(l+1)$ (Ji & Held 2008), we write C_{ei}^0 as

$$C_{ei}^0 = - \sum_{l,k} \frac{n_i L_{ei}}{8\pi c_e^3} \frac{l(l+1) f_{eM}}{\sqrt{\sigma_k^l}} L_k^{l+1/2}(c_e^2) \mathbf{P}^l(\mathbf{c}_e) \cdot \mathbf{M}_e^{lk}. \quad (3.41)$$

Similarly to like-species collisions, we approximate $\mathbf{M}_e^{lk} \simeq \mathbf{M}_{e0}^{lk}$ in (3.41), representing C_{ei}^0 accurately up to $O(\epsilon_v \epsilon_e)$. Using the basis transformation in (3.31) and the gyroaverage property of $\mathbf{P}^l(\mathbf{c}_a)$ in (3.26), we take guiding-centre moments of C_{ei} of the form (3.38), and obtain

$$C_{ei}^{pj} = - \frac{v_{ei}}{8\pi^{3/2}} \sum_{l=0}^{p+2j} \sum_{f=0}^{j+[p/2]} \frac{(T_e^{-1})_{pj}^{lf}}{\sqrt{2^p p!}} \left[\sum_{k=0}^{\infty} A_{ei}^{lf,k} \mathcal{N}_e^{lk} - \delta_{l,1} \frac{u_{\parallel i} - u_{\parallel e}}{v_{the}} \frac{8}{3} \frac{\Gamma(f+3/2)}{f! \sqrt{\pi}} \right], \quad (3.42)$$

where the A_{ei} coefficients are given by

$$A_{ei,0}^{lf,k} = \frac{l(l+1)}{l+1/2} |\mathbf{P}^l(\mathbf{b})|^2 \sum_{m=0}^f \sum_{n=0}^k \frac{L_{fm}^l L_{kn}^l}{\sqrt{\sigma_k^l}} (l+m+n-1)!. \quad (3.43)$$

Finally, for the ion-electron collision operator, C_{ie} , we neglect $O(\sqrt{m_e/m_i} \epsilon_v \epsilon_e)$ corrections by approximating $F_i \simeq \langle F_i \rangle$, and use the transformation in (2.3) to convert the C_{ie} operator in (2.27) to guiding-centre variables, yielding

$$C_{ie} = \frac{\mathbf{R}_{ei}}{m_i n_i v_{thi}} \cdot \left[\mathbf{c}_\perp \frac{m_i v_{thi}^2}{B} \frac{\partial \langle F_i \rangle}{\partial \mu} + \mathbf{b} \frac{\partial \langle F_i \rangle}{\partial c_{\parallel i}} \right] + v_{ei} \frac{m_e n_e}{m_i n_i} \\ \times \left[3 \langle F_i \rangle + c_{\parallel i} \frac{\partial \langle F_i \rangle}{\partial c_{\parallel i}} + 2\mu \frac{\partial \langle F_i \rangle}{\partial \mu} + \frac{T_e}{2T_i} \frac{\partial^2 \langle F_i \rangle}{\partial c_{\parallel i}^2} + \frac{2T_e}{B} \frac{\partial}{\partial \mu} \left(\mu \frac{\partial \langle F_i \rangle}{\partial \mu} \right) \right]. \quad (3.44)$$

By evaluating $\mathbf{R}_{ei}$ at the guiding-centre position $\mathbf{R}$, we write $\mathbf{R}_{ei} \cdot \mathbf{b} = N_e m_e v_{th\parallel e} C_{ei}^{10} / \sqrt{2} + O(\sqrt{m_e/m_i} \epsilon_v \epsilon)$ and gyroaverage (3.44), yielding

$$\langle C_{ie} \rangle = \frac{C_{ei}^{10}}{\sqrt{2}} \frac{m_e N_e v_{th\parallel e}}{m_i n_i v_{th\parallel i}} \frac{\partial \langle F_i \rangle}{\partial s_{\parallel}} + v_{ei} \frac{m_e n_e}{m_i n_i} \\ \times \left[3 \langle F_i \rangle + s_{\parallel i} \frac{\partial \langle F_i \rangle}{\partial s_{\parallel i}} + 2\mu \frac{\partial \langle F_i \rangle}{\partial \mu} + \frac{T_e}{2T_{\parallel i}} \frac{\partial^2 \langle F_i \rangle}{\partial s_{\parallel i}^2} + \frac{2T_e}{B} \frac{\partial}{\partial \mu} \left(\mu \frac{\partial \langle F_i \rangle}{\partial \mu} \right) \right], \quad (3.45)$$

where we used $c_{\parallel i}^2 = s_{\parallel i}^2 T_{\parallel i} / T_i$. Taking guiding-centre moments of the form (3.38) of $\langle C_{ie} \rangle$ in (3.45), we obtain

$$C_{ie}^{pj} = v_{ei} \frac{m_e}{m_i} \sum_{lk} B_{lk}^{pj} N_i^{lk}, \quad (3.46)$$

with

$$B_{lk}^{pj} = 2j\delta_{lp}\delta_{kj-1} \left(1 - \frac{T_e}{T_{\perp i}} \right) - \sqrt{p} \frac{v_{th\parallel e}}{v_{th\parallel i}} \frac{C_{ei}^{10}}{v_{ei}} \delta_{lp-1} \delta_{kj} \\ - (p+2j)\delta_{lp}\delta_{kj} + \sqrt{p(p-1)}\delta_{lp-2}\delta_{kj} \left(\frac{T_e}{T_{\parallel i}} - 1 \right). \quad (3.47)$$

4. Moment hierarchy

In this section, we derive a set of equations that describe the evolution of the guiding-centre moments N_a^{pj} , by integrating in guiding-centre velocity space the conservative form of the Boltzmann equation, (2.32), with the weights $H_p(s_{\parallel a})L_j(s_{\perp a}^2)$. First, we highlight the dependence of $\dot{\mathbf{R}}$ and $\dot{v}_{\parallel}$ on $s_{\parallel a}$ and $s_{\perp a}^2$ by rewriting the equations of motion as

$$\dot{\mathbf{R}} = \mathbf{U}_{0a} + \mathbf{U}_{pa}^* + s_{\perp a}^2 \mathbf{U}_{\nabla Ba}^* + s_{\parallel a}^2 \mathbf{U}_{ka}^* + s_{\parallel a} (v_{th\parallel a} \mathbf{b} + \mathbf{U}_{pa}^{*th}), \quad (4.1)$$

and

$$m_a \dot{v}_{\parallel} = F_{\parallel a} - s_{\perp a}^2 F_{Ma} + s_{\parallel a} F_{pa}^{th} - m_a \mathcal{A}. \quad (4.2)$$

In (4.1), (4.2), $\mathbf{U}_{0a} = \mathbf{v}_E + u_{\parallel a} \mathbf{b}$ is the lowest-order guiding-centre fluid velocity, $\mathbf{U}_{\nabla Ba}^* = (T_{\perp a}/m_a)(\mathbf{b} \times \nabla B/\Omega_a^* B)$ is the fluid grad-B drift, with $\Omega_a^* = q_a B^*/m_a$, $\mathbf{U}_{ka}^* = (2T_{\parallel a}/m_a)(\mathbf{b} \times \mathbf{k}/\Omega_a^*)$ is the fluid curvature drift with $\mathbf{k} = \mathbf{b} \cdot \nabla \mathbf{b}$, $\mathbf{U}_{pa}^* = (\mathbf{b}/\Omega_a^*) \times d_0 \mathbf{U}_{0a}/dt$ is the fluid polarization drift, $F_{\parallel a} = q_a E_{\parallel} + m_a \mathbf{v}_E \cdot d_0 \mathbf{b}/dt$, $F_{Ma} = T_{\perp a} \nabla_{\parallel} \ln B$ is the mirror force, and both $\mathbf{U}_{pa}^{*th}$ and F_{pa}^{th} are related to gradients of the electromagnetic fields

$$\mathbf{U}_{pa}^{*th} = v_{th\parallel a} \frac{\mathbf{b}}{\Omega_a^*} \times (\mathbf{b} \cdot \nabla \mathbf{v}_E + \mathbf{v}_E \cdot \nabla \mathbf{b} + 2u_{\parallel a} \mathbf{k}), \quad (4.3)$$

$$\mathbf{F}_{pa}^{th} = m_a v_{th\parallel a} \mathbf{b} \cdot \left(\frac{\mathbf{k} \times \mathbf{E}}{B} \right). \quad (4.4)$$

The fluid-convective derivative operator is defined as

$$\frac{d_{0a}}{dt} = \partial_t + \mathbf{U}_{0a} \cdot \nabla. \quad (4.5)$$

Next, to obtain an equation for the moment N_a^{pj} , we apply the guiding-centre moment operator

$$\begin{aligned} \|\chi\|_a^{*pj} &= \frac{1}{N_a B} \|\chi H_p(s_{\parallel a}) L_j(s_{\perp a}^2) B_{\parallel}^*\| \\ &= \frac{1}{N_a} \int \chi \frac{B_{\parallel}^*}{m_a} \langle F_a \rangle \frac{H_p(s_{\parallel a}) L_j(s_{\perp a}^2)}{\sqrt{2^p p!}} dv_{\parallel} d\mu d\theta, \end{aligned} \quad (4.6)$$

to Boltzmann's equation, (2.32). By defining $\|1\|_a^{*pj} = \bar{N}_a^{pj}$ such that

$$\begin{aligned} \bar{N}_a^{pj} &= N_a^{pj} \left(1 + \frac{\mathbf{b} \cdot \nabla \times \mathbf{v}_E}{\Omega_a} + u_{\parallel a} \frac{\mathbf{b} \cdot \nabla \times \mathbf{b}}{\Omega_a} \right) \\ &\quad + v_{th\parallel a} \frac{\mathbf{b} \cdot \nabla \times \mathbf{b}}{\sqrt{2}\Omega_a} \left(\sqrt{p+1} N_a^{p+1-j} + \sqrt{p} N_a^{p-1-j} \right), \end{aligned} \quad (4.7)$$

and

$$\frac{d_a^{*pj}}{dt} = \bar{N}_a^{pj} \frac{\partial}{\partial t} + \|\dot{\mathbf{R}}\|_a^{*pj} \cdot \nabla, \quad (4.8)$$

the drift-kinetic moment hierarchy conservation equation for species a is

$$\frac{\partial \bar{N}_a^{pj}}{\partial t} + \nabla \cdot \|\dot{\mathbf{R}}\|_a^{*pj} - \frac{\sqrt{2p}}{v_{th\parallel a}} \|\dot{v}_{\parallel}\|_a^{*p-1j} + \mathcal{F}_a^{pj} = \sum_b C_{ab}^{pj}, \quad (4.9)$$

where we identify the fluid operator

$$\begin{aligned} \mathcal{F}_a^{pj} &= \frac{d_a^{*pj}}{dt} \ln \left(N_a T_{\parallel a}^{p/2} T_{\perp a}^j B^{-j} \right) + \frac{\sqrt{2p}}{v_{th\parallel a}} \frac{d^{*p-1j} u_{\parallel a}}{dt} \\ &\quad + \frac{\sqrt{p(p-1)}}{2} \frac{d_a^{*p-2j}}{dt} \ln T_{\parallel a} - j \frac{d_a^{*pj-1}}{dt} \ln \left(\frac{T_{\perp a}}{B} \right), \end{aligned} \quad (4.10)$$

since it is the key item that describes the evolution of the guiding-centre fluid properties N_a , $u_{\parallel a}$, $P_{\perp a}$, and $P_{\parallel a}$ (see (6)).

The guiding-centre moments of the particle's equations of motion are given by

$$\begin{aligned} \|\dot{\mathbf{R}}\|_a^{*pj} &= \sum_{l,k} \left(U_{0a} \delta_{pl} \delta_{jk} + v_{th\parallel a} \mathbf{b} \mathcal{V}_{lk}^{lpj} \right) \bar{N}_a^{lk} \\ &\quad + \left(U_{pa} \delta_{pl} \delta_{jk} + U_{pa}^{th} \mathcal{V}_{lk}^{lpj} + U_{\nabla Ba} \mathcal{M}_{lk}^{pj} + U_{ka} \mathcal{V}_{lk}^{2pj} \right) N_a^{lk}, \end{aligned} \quad (4.11)$$

$$m_a \|\dot{v}_{\parallel}\|_a^{*pj} = \sum_{l,k} \left[F_{\parallel a} \delta_{p,l} \delta_{j,k} + F_{pa}^{th} \mathcal{V}_{lk}^{lpj} + F_{Ma} \mathcal{M}_{lk}^{pj} \right] \bar{N}_a^{lk} + m_a \|\mathcal{A}\|_a^{*pj}, \quad (4.12)$$

where the phase-mixing operators read

$$\mathcal{V}_{lk}^{1pj} = \left(\sqrt{\frac{p+1}{2}} \delta_{p+1,l} + \sqrt{\frac{p}{2}} \delta_{p-1,l} \right) \delta_{k,j}, \quad (4.13)$$

$$\mathcal{V}_{lk}^{2pj} = \left[\delta_{p,l} \left(p + \frac{1}{2} \right) + \frac{\sqrt{(p+2)(p+1)}}{2} \delta_{p+2,l} + \frac{\sqrt{p(p-1)}}{2} \delta_{p-2,l} \right] \delta_{j,k}, \quad (4.14)$$

$$\mathcal{M}_{lk}^{pj} = (2j+1) \delta_{p,l} \delta_{j,k} - (j+1) \delta_{p,l} \delta_{j+1,k} - j \delta_{p,l} \delta_{j-1,k}. \quad (4.15)$$

The expressions of U_{pa} , $U_{\nabla Ba}$, U_{pa}^{th} and U_{ka} are derived from U_{pa}^* , $U_{\nabla Ba}^*$, U_{pa}^{*th} and U_{ka}^* by replacing Ω_a^* with Ω_a . The expression of $\|\mathcal{A}\|^{*pj}$ can be found in appendix B.

Similar moment hierarchy models (with uniform magnetic fields) have been numerically implemented, and successfully compared with their kinetic counterpart (Paškauskas & De Ninno 2009; Loureiro *et al.* 2016; Schekochihin *et al.* 2016; Grošelj *et al.* 2017), and even shown to be more efficient than other velocity discretization techniques in the same region of validity (Camporeale *et al.* 2016). Equation (4.9) generalizes such models to spatially varying fields and full Coulomb collisions, while retaining phase-mixing operators that couple nearby Hermite and Laguerre moments and providing a close form for the projection of the Coulomb operator in velocity space. We also note that the use of shifted velocity polynomials in the Hermite–Laguerre basis, which gives rise to the fluid operator $\mathcal{F}_a^{pj}$, allows us to have an efficient representation of the distribution function both in the weak ($u_{\parallel a} \ll v_{tha}$) and strong flow ($u_{\parallel a} \sim v_{tha}$) regimes. As we will see in § 6, the fluid operator $\mathcal{F}_a^{pj}$ generates the lowest-order fluid equations, as it is present even if all kinetic moments N_a^{pj} (except N_a^{00}) are set to zero.

5. Poisson's equation

We use Poisson's equation to evaluate the electric field appearing in the moment hierarchy equation, equation (4.9). In $(\mathbf{x}, \mathbf{v})$ coordinates, Poisson's equation reads

$$\epsilon_0 \nabla \cdot \mathbf{E} = \sum_a q_a n_a = \sum_a q_a \int f_a d^3 v. \quad (5.1)$$

Following the same steps used to derive (3.24) from (3.20), we can write Poisson's equation, equation (5.1), as

$$\epsilon_0 \nabla \cdot \mathbf{E} = \sum_a q_a \int d^3 \mathbf{R} dv_{\parallel} d\mu d\theta \frac{B_{\parallel}^*}{m} \delta(\mathbf{R} + \rho_a \mathbf{a} - \mathbf{x}) F_a(\mathbf{R}, v_{\parallel}, \mu, \theta). \quad (5.2)$$

Equation (5.2) shows that all particles that have a Larmor orbit crossing a given point $\mathbf{x}$, give a contribution to the charge density at this location.

Performing the integral over $\mathbf{R}$ and introducing the Fourier transform $F_a(\mathbf{x} - \rho_a \mathbf{a}, v_{\parallel}, \mu, \theta) = \int d^3 \mathbf{k} F_a(\mathbf{k}, v_{\parallel}, \mu, \theta) e^{-i\mathbf{k} \cdot \mathbf{x}} e^{i\rho_a \mathbf{k} \cdot \mathbf{a}}$, equation (5.2) can be rewritten as

$$\epsilon_0 \nabla \cdot \mathbf{E} = \sum_a q_a \int dv_{\parallel} d\mu d^3 \mathbf{k} d\theta \frac{B_{\parallel}^*}{m_a} F_a(\mathbf{k}, v_{\parallel}, \mu, \theta) e^{-i\mathbf{k} \cdot \mathbf{x}} e^{i\rho_a \mathbf{k} \cdot \mathbf{a}}. \quad (5.3)$$

To perform the $\mathbf{k}$ integration, we use the cylindrical coordinate system $(k_\perp, \alpha, k_\parallel)$, expressing $\mathbf{k} = k_\perp(\cos \alpha \mathbf{e}_1 + \sin \alpha \mathbf{e}_2) + k_\parallel \mathbf{b}$, such that $\mathbf{k} \cdot \mathbf{a} = k_\perp \cos(\theta - \alpha)$. This coordinate system allows us to express $e^{i\rho_a \mathbf{k} \cdot \mathbf{a}}$ in (5.3) in terms of Bessel functions. Indeed, $e^{ik_\perp \rho_a \cos(\theta - \alpha)} = J_0(k_\perp \rho_a) + 2 \sum_{l=1}^{\infty} J_l(k_\perp \rho_a) i^l \cos[l(\theta - \alpha)]$ (Abramowitz, Stegun & Miller 1965), where $J_l(k_\perp \rho_a)$ is the Bessel function of the first kind of order l . We can then write

$$\epsilon_0 \nabla \cdot \mathbf{E} = \sum_a q_a \int dv_\parallel d\mu d\theta \frac{B_\parallel^*}{m_a} \left(\Gamma_0[F_a] + 2 \sum_{l=1}^{\infty} i^l \Gamma_l[F_a \cos[l(\theta - \alpha)]] \right), \quad (5.4)$$

where the Fourier–Bessel operator $\Gamma_l[f]$ is defined as

$$\Gamma_l[F_a(\mathbf{k}, v_\parallel, \mu, \theta)] \equiv \int d^3\mathbf{k} J_l(k_\perp \rho_a) F_a(\mathbf{k}, v_\parallel, \mu, \theta) e^{-i\mathbf{k} \cdot \mathbf{x}}. \quad (5.5)$$

Introducing the Fourier decomposition of $\tilde{F}_a$, equation (2.39), in (5.4), we obtain

$$\epsilon_0 \nabla \cdot \mathbf{E} = \sum_a q_a \int dv_\parallel d\mu \frac{B_\parallel^*}{m} \left[\Gamma_0[\langle F_a \rangle] + 2\pi \sum_{l=1}^{\infty} \frac{i^{l-1}}{l\Omega_a} \Gamma_l[C_{la} e^{il\alpha} + C_{-la} e^{-il\alpha}] \right], \quad (5.6)$$

where the θ integration was performed by using the identity $\int_0^{2\pi} e^{i\theta(l-m)} d\theta = 2\pi \delta(l-m)$. Notice that $\int_0^{2\pi} \Gamma_0[F_a] d\theta / 2\pi = \Gamma_0(\langle F_a \rangle)$, and corresponds to the $J_0(k_\perp \rho_a)$ operator used in most gyrofluid closures (Hammett, Dorland & Perkins 1992; Snyder & Hammett 2001; Madsen 2013), and in the gyrokinetic Poisson equation (Dubin, Krommes & Oberman 1983; Lee 1983).

We now order the terms appearing in (5.6). Using the Taylor series expansion of a Bessel function $J_l(x)$ of order l (Abramowitz *et al.* 1965), we find

$$\Gamma_0[\langle F_a \rangle] \sim \left[1 - \frac{(k_\perp \rho_a)^2}{4} + O(\epsilon^4) \right] \langle F_a \rangle, \quad (5.7)$$

while using the orderings of v_e and v_i in (1.2) and (2.28)

$$\frac{\Gamma_l[C_{la}]}{\Omega_a} \lesssim \epsilon_v \epsilon^{l+1} \langle F_a \rangle \quad (5.8)$$

for $l \geq 1$.

Consistently with § 3.3, we neglect the $l \geq 1$ collisional terms, therefore representing Poisson's equation up to $O(\epsilon_v \epsilon)$. For the derivation of an higher-order Poisson equation, the treatment of finite $l \geq 1$ collisional effects are presented in appendix C. Taylor expanding $J_0(x) \simeq 1 - x^2/4$, Poisson's equation reads

$$\epsilon_0 \nabla \cdot \mathbf{E} = \sum_a q_a \left[N_a \left(1 + \frac{\mathbf{b} \cdot \nabla \times \mathbf{b}}{\Omega_a} u_{\parallel a} + \frac{\mathbf{b} \cdot \nabla \times \mathbf{v}_E}{\Omega_a} \right) + \frac{1}{2m_a} \nabla_\perp^2 \left(\frac{P_{\perp a}}{\Omega_a^2} \right) \right]. \quad (5.9)$$

6. Collisional drift-reduced fluid model

The infinite set of equations that describe the evolution of the moments of the distribution function, equation (4.9), and Poisson's equation, equation (5.9), constitute the drift-reduced model, which is valid for distribution functions arbitrarily far from equilibrium. For practical purposes, a closure scheme must be provided in order to reduce the model to a finite number of equations. In this section, we derive a closure in the high-collisionality regime. For this purpose, we first state in § 6.1 the evolution equations for the fluid moments (i.e. n_a , $u_{\parallel a}$, $T_{\parallel a}$, $T_{\perp a}$, $Q_{\parallel a}$ and $Q_{\perp a}$), that correspond to the lowest-order indices of the moment hierarchy equation. Then, in § 6.2, we apply a prescription for the higher-order parallel and perpendicular moment equations that allows a collisional closure for $Q_{\parallel a}$ and $Q_{\perp a}$ in terms of n_a , $u_{\parallel a}$, $T_{\parallel a}$ and $T_{\perp a}$. The nonlinear closure prescription used here, sometimes called semi-collisional closure (Zocco & Schekochihin 2011), can be employed at arbitrary collisionalities by including a sufficiently high number of moments (indeed, it was used in Loureiro *et al.* (2016) to consider low-collisionality regimes). It also allows us to retain the nonlinear collision contributions inherent to a full-F description that may have the same size as its linear contributions, as pointed out in Catto & Simakov (2004).

6.1. Fluid equations

We first look at the $(p, j) = (0, 0)$ case of (4.9). Noting that $\bar{N}_a^{00} = 0$ and $C_{ab}^{00} = 0$, we obtain

$$\nabla \cdot \|\dot{\mathbf{R}}\|_a^{*00} + \mathcal{F}_a^{00} = 0. \quad (6.1)$$

Evaluating $\|\dot{\mathbf{R}}\|_a^{*pj}$ in (4.11) and $\mathcal{F}_a^{pj}$ in (4.10), for $(p, j) = (0, 0)$, equation (6.1) yields the continuity equation

$$\frac{d_a^0 N_a}{dt} + \frac{d_{0a}}{dt} \left(\frac{N_a \nabla_{\perp}^2 \phi}{\Omega_a B} \right) = -N_a \nabla \cdot \mathbf{u}_{0a} - \frac{N_a \nabla_{\perp}^2 \phi}{\Omega_a B} \nabla \cdot \mathbf{U}_{0a}. \quad (6.2)$$

The upper-convective derivative d_a^0/dt , defined by

$$\frac{d_a^0}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_{0a} \cdot \nabla, \quad (6.3)$$

is related to the guiding-centre fluid velocity $\mathbf{u}_{0a}$

$$\mathbf{u}_{0a} = \mathbf{U}_{0a} + \frac{T_{\parallel a} + T_{\perp a}}{m_a} \frac{\mathbf{b} \times \nabla B}{\Omega_a B} + \frac{\mathbf{b}}{\Omega_a} \times \frac{d_{0a} \mathbf{U}_{0a}}{dt}, \quad (6.4)$$

and it differs from the lower-convective derivative d_{0a}/dt in (4.5) by the addition of the last two terms in (6.4). The vorticity $\nabla_{\perp}^2 \phi$ is related to the $\mathbf{E} \times \mathbf{B}$ drift by

$$\frac{\mathbf{b} \cdot \nabla \times \mathbf{v}_E}{\Omega_a} = \frac{\nabla_{\perp}^2 \phi}{B \Omega_a} + O(\epsilon^3), \quad (6.5)$$

and it appears in (6.2) due to the difference between $\bar{N}_a^{00}$ and N_a^{00} [see (4.7)]. To derive (6.2), we use the low- β limit expression for $\mathbf{b} \times \mathbf{k} \simeq (\mathbf{b} \times \nabla B)/B$ and

neglect $u_{\parallel a} \mathbf{b} \cdot \nabla \times \mathbf{b} / \Omega_a$ as

$$\frac{u_{\parallel a} \mathbf{b} \cdot \nabla \times \mathbf{b}}{\Omega_a} \sim \frac{T_e}{T_i} \beta \sim \epsilon^3, \quad (6.6)$$

therefore keeping up to $O(\epsilon^2)$ terms (namely the $\nabla_{\perp}^2 \phi$ term in (6.5)).

The parallel momentum equation is obtained by setting $(p, j) = (1, 0)$ in (4.9), yielding

$$\begin{aligned} m_a \frac{d^0 u_{\parallel a}}{dt} = & \frac{m_a v_{th\parallel a}}{\sqrt{2}} \sum_b C_{ab}^{10} - \frac{m_a \nabla_{\perp}^2 \phi}{\Omega_a B} \frac{d_0 u_{\parallel a}}{dt} - \frac{m_a}{\sqrt{2} N_a} \nabla \cdot (\mathbf{u}_a^1 N_a v_{th\parallel a}) \\ & + m_a \|\mathcal{A}\|_a^{*00} + \left(1 + \frac{\nabla_{\perp}^2 \phi}{\Omega_a B}\right) \left(q_a E_{\parallel} - T_{\perp a} \frac{\nabla_{\parallel} B}{B} + m_a \mathbf{v}_E \cdot \frac{d_0 \mathbf{a}}{dt}\right), \end{aligned} \quad (6.7)$$

with

$$\mathbf{u}_a^1 = \frac{\mathbf{U}_{pa}^{th}}{\sqrt{2}} + \frac{\sqrt{2} \mathbf{b} \times \nabla B}{m_a} \frac{Q_{\parallel a} + Q_{\perp a}}{\Omega_a B} + v_{th\parallel a} \frac{\mathbf{b}}{2} \left(1 + \frac{\nabla_{\perp}^2 \phi}{\Omega_a B}\right). \quad (6.8)$$

The expression for C_{ab}^{10} is given in appendix D, as well as all the C_{ab}^{pj} coefficients relevant for the present fluid model. The left-hand side of (6.7) describes the convection of $u_{\parallel a}$, while the first term in the right-hand side is related to pressure and heat flux gradients, the second term to resistivity (collisional effects), the third term consists of high-order terms kept to ensure phase-space conservation properties, and the last term is the parallel fluid acceleration, namely due to parallel electric fields, mirror force and inertia.

The parallel and perpendicular temperature equations are obtained by setting $(p, j) = (2, 0)$ and $(0, 1)$ respectively in (4.9). This yields for the parallel temperature

$$\begin{aligned} \frac{N_a}{\sqrt{2}} \frac{d^0 T_{\parallel a}}{dt} = & \sqrt{2} Q_{\perp a} \frac{\nabla_{\parallel} B}{B} - \frac{N_a \nabla_{\perp}^2 \phi}{\sqrt{2} \Omega_a B} \frac{d_0 T_{\parallel a}}{dt} - 2 \frac{N_a T_{\parallel a}}{v_{th\parallel a}} \mathbf{u}_a^1 \cdot \nabla u_{\parallel a} \\ & - \nabla \cdot (N_a T_{\parallel a} \mathbf{u}_a^{2\parallel}) + N_a T_{\parallel a} \frac{\mathbf{E}}{B} \cdot \frac{\mathbf{b} \times \nabla B}{B} \left(1 + \frac{\nabla_{\perp}^2 \phi}{\Omega_a B}\right) \\ & + \sum_b C_{ab}^{20} N T_{\parallel a} + \frac{2 N_a T_{\parallel a}}{v_{th\parallel a}} \|\mathcal{A}\|_a^{*10}, \end{aligned} \quad (6.9)$$

where

$$\mathbf{u}_a^{2\parallel} = \frac{Q_{\parallel a}}{2 N_a T_{\parallel a}} \frac{\mathbf{U}_{pa}^{th}}{v_{th\parallel a}} + \frac{\sqrt{2} T_{\parallel a}}{m_a} \frac{\mathbf{b} \times \nabla B}{\Omega_a B} + \frac{\mathbf{b}}{2} \frac{Q_{\parallel a}}{N_a T_{\parallel a}} \left(1 + \frac{\nabla_{\perp}^2 \phi}{\Omega_a B}\right), \quad (6.10)$$

and for the perpendicular temperature

$$N_a \frac{d^0}{dt} \left(\frac{T_{\perp a}}{B}\right) + \frac{N_a \nabla_{\perp}^2 \phi}{\Omega_a B} \frac{d_0}{dt} \left(\frac{T_{\perp a}}{B}\right) = \nabla \cdot \left(\frac{N_a T_{\perp a}}{B} \mathbf{u}_a^{2\perp}\right) - \frac{N_a T_{\perp a}}{B} \sum_b C_{ab}^{01}, \quad (6.11)$$

with

$$\mathbf{u}_a^{2\perp} = -\frac{Q_{\perp a}}{N_a T_{\perp a}} \frac{\mathbf{U}_{pa}^{th}}{v_{th\parallel a}} - \frac{T_{\perp a}}{m_a} \frac{\mathbf{b} \times \nabla B}{\Omega_a B}. \quad (6.12)$$

The equations for the evolution of the parallel $Q_{\parallel a}$ and perpendicular $Q_{\perp a}$ heat fluxes are obtained by setting $(p, j) = (3, 0)$ and $(1, 1)$ respectively in (4.9), yielding

$$\begin{aligned} \frac{d_a^0 Q_{\parallel a}}{dt} = & -\frac{d_{0a}}{dt} \left(Q_{\parallel a} \frac{\nabla_{\perp}^2 \phi}{\Omega_a B} \right) + N_a T_{\parallel a} \sqrt{3} v_{tha} \sum_b C_{ab}^{30} \\ & - Q_{\parallel a} \nabla \cdot \mathbf{u}_a^0 - \frac{Q_{\parallel a} \nabla_{\perp}^2 \phi}{\Omega_a B} \nabla \cdot \mathbf{U}_{0a} - 3 \nabla \cdot (\mathbf{u}_{ka} Q_{\parallel a}) \\ & - \frac{3}{\sqrt{2}} \left(1 + \frac{\nabla_{\perp}^2 \phi}{\Omega_a B} \right) \frac{\mathbf{E} \cdot \mathbf{b} \times \nabla B}{B^2} Q_{\parallel a} + 3\sqrt{2} N_a T_{\parallel a} \|\mathcal{A}_{\parallel a}\|^{*20} \\ & - 3\sqrt{2} N_a T_{\parallel a} \mathbf{u}_a^{2\parallel} \cdot \nabla u_{\parallel a} - 3\sqrt{2} N_a v_{tha} \mathbf{u}_a^1 \cdot \nabla T_{\parallel a}, \end{aligned} \quad (6.13)$$

and

$$\begin{aligned} \frac{d_a^0}{dt} \left(\frac{Q_{\perp a}}{B} \right) = & -\frac{d_{0a}}{dt} \left(\frac{Q_{\perp a}}{B} \frac{\nabla_{\perp}^2 \phi}{\Omega_a B} \right) - \frac{N_a v_{tha}}{\sqrt{2}} (\mathbf{u}_a^1 \cdot \nabla) \frac{T_{\perp a}}{B} \\ & + \frac{N_a T_{\perp a}}{B} (\mathbf{u}_a^{2\perp} \cdot \nabla) u_{\parallel a} - \left(\frac{Q_{\perp a}}{B} \right) \left(\nabla \cdot \mathbf{u}_a^0 + \frac{\nabla_{\perp}^2 \phi}{\Omega_a B} \nabla \cdot \mathbf{U}_{0a} \right) \\ & - (\mathbf{U}_{ka} + 2\mathbf{U}_{\nabla B}) \cdot \nabla \left(\frac{Q_{\perp a}}{B} \right) - \frac{\sum_b C_{ab}^{11} v_{tha} N_a T_{\perp a}}{\sqrt{2} B} \\ & + \left(\frac{N_a T_{\perp a}^2}{m_a} \frac{\nabla_{\parallel} B}{B^2} + \frac{Q_{\perp a}}{B} \mathbf{E} \cdot \frac{\mathbf{b} \times \nabla B}{B^2} \right) \left(1 + \frac{\nabla_{\perp}^2 \phi}{\Omega_a B} \right). \end{aligned} \quad (6.14)$$

In (6.13) and (6.14) we neglected the higher-order moments with respect to N^{30} and N^{11} , an approximation that we will scrutinize in the next section. Equations (6.2)–(6.14) constitute a closed set of six coupled nonlinear partial differential equations for both the hydrodynamical variables n_a , $u_{\parallel a}$, $T_{\parallel a}$, $T_{\perp a}$ and the kinetic variables $Q_{\parallel a}$ and $Q_{\perp a}$.

With respect to previous delta-F (Brizard 1992; Dorland & Hammett 1993) and full-F gyrofluid models (Madsen 2013), our fluid model, equations (6.2)–(6.14), while neglecting $k_{\perp} \rho_i \sim 1$ effects, includes the velocity contributions from the $B_{\parallel}^*$ denominator in the equations of motion (2.15) and (2.16) and includes the effects of full Coulomb collisions up to order $\epsilon_v \epsilon$. Also, due to the choice of basis functions with shifted velocity arguments $H_p(s_{\parallel a})$ instead of $H_p(v_{\parallel}/v_{tha})$, we obtain a set of equations that can efficiently describe both weak flow ($u_{\parallel a} \ll v_{tha}$) and strong flow ($u_{\parallel a} \sim v_{tha}$) regimes.

6.2. High-collisionality regime

We now consider the high-collisionality regime, where the characteristic fluctuation frequency of the hydrodynamical variables ω

$$\omega \sim v_{tha} |\nabla_{\parallel} \ln N_a| \sim v_{tha} |\nabla_{\parallel} \ln T_{\parallel a}| \sim v_{tha} |\nabla_{\parallel} \ln T_{\perp a}| \sim |\nabla_{\parallel} u_{\parallel a}| \sim v_{tha}/L_{\parallel a}, \quad (6.15)$$

is much smaller than the collision frequency $\nu_a \simeq \nu_{aa}$, that is

$$\delta_a \sim \frac{\omega}{\nu_a} \sim \frac{\lambda_{mfpa}}{L_{\parallel a}} \ll 1, \quad (6.16)$$

where the mean free path λ_{mfpa} in (6.16) is defined as

$$\lambda_{mfpa} = v_{tha}/\nu_{aa}. \quad (6.17)$$

Equation (6.16) describes the so-called linear transport regime (Balescu 1988). In this case, the distribution function can be expanded around a Maxwell–Boltzmann equilibrium, according to the Chapman–Enskog asymptotic closure scheme (Chapman 1962) and, to first order in δ_a , we have

$$\langle F_a \rangle \simeq F_{Ma}[1 + \delta_a f_{1a}(v_{\parallel}, \mu, \mathbf{R}, t)]. \quad (6.18)$$

According to (6.18), all moments N_a^{pj} in the Hermite–Laguerre expansion (3.1) with $(p, j) \neq (0, 0)$ are order δ_a . Since $Q_{\parallel a}$ and $Q_{\perp a}$ are determined at first order in δ_a only by the moments $(p, j) = (0, 0), (3, 0), (1, 1)$, the truncation of Sec. (6.1), i.e. neglecting $(p, j) \neq (0, 0), (3, 0), (1, 1)$ is justified. For a more detailed discussion on this topic see Balescu (1988). Moreover, in the linear regime, a relationship between the hydrodynamical and kinetic variables can be obtained along the lines of the semi-collisional closure. This allows us to express $Q_{\parallel a}$ and $Q_{\perp a}$ as a function of $N_a, u_{\parallel a}, T_{\parallel a}$ and $T_{\perp a}$, therefore reducing the number of equations. We now derive this functional relationship.

We consider (6.13)–(6.14) in the linear regime, and neglect the polarization terms that are proportional to $\nabla_{\perp}^2 \phi / (\Omega_a B)$. This yields $\sqrt{3/2} \sum_b C_{ab}^{30} / v_{th\parallel a} \simeq R_{\parallel a}$ and $\sum_b C_{ab}^{11} / (\sqrt{2} v_{th\parallel a}) \simeq R_{\perp a}$, with $R_{\parallel a}$ and $R_{\perp a}$ given by

$$R_{\parallel a} = \frac{\nabla_{\parallel} T_{\parallel a}}{T_{\parallel a}} + u_{\parallel a} \frac{\mathbf{b} \times \nabla B}{\Omega_a B} \cdot \left(\frac{\nabla u_{\parallel a}}{u_{\parallel a}} + \frac{\nabla T_{\parallel a}}{T_{\parallel a}} \right), \quad (6.19)$$

$$R_{\perp a} = \frac{T_{\perp a}}{T_{\parallel a}} \frac{\nabla_{\parallel} B}{B} - \frac{1}{2\sqrt{2}} \nabla_{\parallel} \ln \frac{T_{\perp a}}{B} - u_{\parallel a} \frac{\mathbf{b} \times \nabla B}{\Omega_a B} \cdot \left(\frac{T_{\perp a}}{T_{\parallel a}} \frac{\nabla u_{\parallel a}}{u_{\parallel a}} + \nabla \ln \frac{T_{\perp a}}{B} \right), \quad (6.20)$$

since $d^0/dt \sim d_{0a}/dt \sim \omega$ and $(d^0 Q_{\parallel, \perp}/dt)/Q_{\parallel, \perp a} \sim \delta_a^2 \nu_a$. We compute the guiding-centre moments of the collision operator C_{ab}^{30} and C_{ab}^{11} by truncating the series for the like-species collision operator in (3.40) at $(l, k, n, q) = (2, 1, 2, 1)$. The resulting C_{ab}^{pj} coefficients are presented in appendix D.

With the expression of C_{ab}^{30} and C_{ab}^{11} , we can solve for $Q_{\parallel a}$ and $Q_{\perp a}$. In the regime $(T_{\parallel a} - T_{\perp a})/T_a \sim \delta$, at lowest order, we obtain for the electron species

$$\frac{Q_{\parallel e}}{N_e T_e v_{the}} = -0.362 \frac{u_{\parallel e} - u_{\parallel i}}{v_{the}} - 10.6 \lambda_{mfpe} \frac{\nabla_{\parallel} T_e}{T_e}, \quad (6.21)$$

and

$$\frac{Q_{\perp e}}{N_e T_e v_{the}} = -0.119 \frac{u_{\parallel e} - u_{\parallel i}}{v_{the}} - 3.02 \lambda_{mfpe} \frac{\nabla_{\parallel} T_e}{T_e}. \quad (6.22)$$

Analogous expressions are obtained for the ion species.

Equations (6.2), (6.7), (6.9) and (6.11), with $Q_{\parallel a}$ and $Q_{\perp a}$ given by (6.21) and (6.22) are valid in the high-collisionality regime, and can be compared with the drift-reduced Braginskii equations in Zeiler *et al.* (1997). We first rewrite the continuity equation, equation (6.2), in the form

$$\frac{\partial N_e}{\partial t} + \nabla \cdot \left[N_e \left(\mathbf{v}_E + u_{\parallel e} \mathbf{b} + \frac{T_{\parallel e} + T_{\perp e}}{m_e} \frac{\mathbf{b} \times \nabla B}{\Omega_e B} \right) \right] = 0, \quad (6.23)$$

where we expand the convective derivative $d^0 a/dt$ using (6.3) and (6.4), and neglect polarization terms proportional to the electron mass m_e . By noting that the diamagnetic drift $\mathbf{v}_{de}$ can be written as

$$\mathbf{v}_{de} = \frac{1}{eN_e} \nabla \times \frac{p_e \mathbf{b}}{B} - 2 \frac{T_e}{m_e} \frac{\mathbf{b} \times \nabla B}{\Omega_e B}, \quad (6.24)$$

and by considering the isotropic regime $T_{\parallel e} \sim T_{\perp e} \sim T_e$, we obtain

$$\frac{\partial N_e}{\partial t} + \nabla \cdot [N_e(\mathbf{v}_E + u_{\parallel e} \mathbf{b} + \mathbf{v}_{de})] = 0, \quad (6.25)$$

which corresponds to the continuity equation in the drift-reduced Braginskii model in Zeiler *et al.* (1997). In that model, the polarization equation is obtained by subtracting both electron and ion continuity equations, using Poisson's equation $n_e \simeq n_i$ with n_e and n_i the electron and ion particle densities respectively, and neglecting the electron to ion mass ratio. Applying the same procedure to the present fluid model, we obtain

$$\begin{aligned} 0 = & \nabla \cdot \left(\frac{\nabla_{\perp}^2 \phi N_i u_{\parallel i} \mathbf{b}}{\Omega_i B} \right) - \nabla \cdot \left[\frac{\mathbf{v}_E}{2m_i} \nabla_{\perp}^2 \left(\frac{N_i T_{\perp i}}{\Omega_i^2} \right) \right] - \frac{1}{2m_i} \frac{\partial}{\partial t} \nabla_{\perp}^2 \left(\frac{N_i T_{\perp i}}{\Omega_i^2} \right) \\ & + \nabla \cdot \left(\frac{N_i}{\Omega_i} \mathbf{b} \times \frac{d_{0i} \mathbf{U}_{0i}}{dt} \right) + \nabla \cdot [\mathbf{b} (N_i u_{\parallel i} - N_e u_{\parallel e})] \\ & + \nabla \cdot \left[(N_i T_{\parallel i} + N_e T_{\parallel e} + N_i T_{\perp i} + N_e T_{\perp e}) \frac{\mathbf{b} \times \nabla B}{eB^2} \right]. \end{aligned} \quad (6.26)$$

In (6.26), the first three terms, which are not present in the drift-reduced Braginskii model, correspond to the difference between ion guiding-centre density N_i and particle density n_i , proportional to both $\nabla_{\perp}^2 \phi$ and $\nabla_{\perp}^2 P_i$. The parallel momentum and temperature equations, equation (6.7) and (6.9), with respect to (Zeiler *et al.* 1997), contain the higher-order term $\mathcal{A}$ that ensures phase-space conservation, mirror force terms proportional to $(\nabla_{\parallel} B)/B$ and polarization terms proportional to $\nabla_{\perp}^2 \phi/(\Omega_a B)$ due to the difference between guiding-centre and particle fluid quantities. This set of fluid equations constitute an improvement over the drift-reduced Braginskii model. With respect to the original Braginskii equations (Braginskii 1965), they include the nonlinear terms that arise when retaining full Coulomb collisions, and the effect of ion–electron collisions.

7. Conclusion

In the present work, a full-F drift-kinetic model is developed, suitable to describe the plasma dynamics in the SOL region of tokamak devices at arbitrary collisionality. Taking advantage of the separation between the turbulent and gyro-motion scales, a gyroaveraged Lagrangian and its corresponding equations of motion are obtained. This is the starting point to deduce a drift-kinetic Boltzmann equation with full Coulomb collisions for the gyroaveraged distribution function.

The gyroaveraged distribution function is then expanded into an Hermite–Laguerre basis, and the coefficients of the expansion are related to the lowest-order gyrofluid moments. The fluid moment expansion of the Coulomb operator described in Ji & Held (2009) is reviewed, and its respective particle moments are written in terms of coefficients of the Hermite–Laguerre expansion, relating both expansions. This allows us to express analytically the moments of the collision operator in terms of guiding-centre moments. A set of equations, indeed a moment hierarchy that describes the evolution of the guiding-centre moments is derived, together with a Poisson's equation accurate up to ϵ^2 . These are then used to derive a fluid model in the high-collisionality limit.

The drift-kinetic model derived herein can be considered a starting point for the development of a gyrokinetic Boltzmann equation suitable for the SOL region (e.g. Qin *et al.* 2007; Hahm, Wang & Madsen 2009). Indeed, using a similar approach, a gyrokinetic moment hierarchy may be derived, allowing for the use of perpendicular wave numbers satisfying $k_\perp \rho_s \sim 1$. For a recent Hermite–Laguerre formulation of the nonlinear delta-F gyrokinetic equation see Mandell *et al.* (2017).

Acknowledgements

This work has been carried out within the framework of the EUROfusion Consortium and has received funding from the Euratom research and training programme 2014–2018 under grant agreement No 633053, and from Portuguese FCT - Fundação para a Ciência e Tecnologia, under grant PD/BD/105979/2014. N.F.L. was partially funded by US Department of Energy Grant no. DE-FG02-91ER54109. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

Appendix A. Basis transformation

In the present appendix, we derive the expressions for the coefficients T_{alk}^{pj} appearing in (3.30). These coefficients allows us to express up to order $\epsilon\epsilon_v$ the relation between fluid M_a^{lk} and guiding-centre N_a^{lk} moments via (3.32). As a first step, we define a transformation similar to (3.30) but with isotropic temperatures between both bases

$$c_a^l P_l(\xi_a) L_k^{l+1/2}(c_a^2) = \sum_{p=0}^{l+2k} \sum_{j=0}^{k+\lfloor l/2 \rfloor} \bar{T}_{lka}^{pj} H_p \left(\frac{v_\parallel - u_{\parallel a}}{v_{tha}} \right) L_j \left(\frac{v_\perp^2}{v_{tha}^2} \right), \quad (\text{A } 1)$$

with the inverse transformation

$$H_p \left(\frac{v_\parallel - u_{\parallel a}}{v_{tha}} \right) L_j \left(\frac{v_\perp^2}{v_{tha}^2} \right) = \sum_{l=0}^{p+2j} \sum_{k=0}^{j+\lfloor p/2 \rfloor} (\bar{T}^{-1})_{pja}^{lk} c_a^l P_l(\xi_a) L_k^{l+1/2}(c_a^2). \quad (\text{A } 2)$$

The relation between the coefficients $(\bar{T}^{-1})_{pja}^{lk}$ and $\bar{T}_{lka}^{pj}$ is given by

$$(\bar{T}^{-1})_{pja}^{lk} = \frac{\sqrt{\pi} 2^p p! (l+1/2) k!}{(k+l+1/2)!} \bar{T}_{lka}^{pj}. \quad (\text{A } 3)$$

By integrating both sides of (A 1) over the whole velocity space, the expression for $\bar{T}_{lk}^{pj}$ is obtained

$$\begin{aligned} \bar{T}_{lk}^{pj} = & \sum_{q=0}^{\lfloor l/2 \rfloor} \sum_{v=0}^{\lfloor p/2 \rfloor} \sum_{i=0}^k \sum_{r=0}^q \sum_{s=0}^{\min(j,i)} \sum_{m=0}^{k-i} \frac{(-1)^{q+i+j+v+m}}{2^{((3l+p)/2)+m+v-r}} \\ & \times \binom{l}{q} \binom{2(l-q)}{l} \binom{q}{r} \binom{r}{j-s} \binom{r}{i-s} \binom{s+r}{s} r! \\ & \times \frac{(k-i+l-1/2)!(l+p+2(m-r-v)-1)!!}{(p-2v)!(k-i-m)!(l+m-1/2)!v!m!}. \end{aligned} \quad (\text{A } 4)$$

We then integrate both sides of (3.30) with weights $H_l(s_{\parallel a})L_j(s_{\perp a}^2)$, with the argument transformation

$$H_p(s_{\parallel a}) = \left(\frac{T_a}{T_{\parallel a}} \right)^{p/2} \sum_{k=0}^{\lfloor p/2 \rfloor} \frac{p!}{k!(p-2k)!} \left(1 - \frac{T_{\parallel a}}{T_a} \right)^k H_{p-2k} \left(\frac{v_{\parallel} - u_{\parallel a}}{v_{tha}} \right), \quad (\text{A } 5)$$

and

$$L_j(s_{\perp a}^2) = \sum_{k=0}^j \binom{j}{j-k} \left(\frac{T_a}{T_{\perp a}} \right)^k \left(1 - \frac{T_a}{T_{\perp a}} \right)^{j-k} L_k \left(\frac{v_{\perp}^{'2}}{v_{tha}^2} \right), \quad (\text{A } 6)$$

to find the relation between the isotropic and anisotropic temperature coefficients

$$\begin{aligned} T_{alk}^{pj} = & \sum_{pp=0}^{l+2k} \sum_{jj=0}^{k+\lfloor l \rfloor 2} \sum_{z=0}^{jj} \sum_{d=0}^{\lfloor pp/2 \rfloor} \binom{jj}{jj-z} \frac{pp! \delta_{z,j} \delta_{p,pp-2d}}{d!(pp-2d)!} \\ & \times \left(\frac{T_{\parallel a}}{T_a} \right)^{p/2} \left(\frac{T_{\perp a}}{T_a} \right)^z \left(1 - \frac{T_a}{T_{\parallel a}} \right)^d \left(1 - \frac{T_{\perp a}}{T_a} \right)^{jj-z} \bar{T}_{lk}^{ppij}, \end{aligned} \quad (\text{A } 7)$$

$$\begin{aligned} (T_a^{-1})_{pj}^{lk} = & \sum_{z=0}^j \sum_{d=0}^{\lfloor p/2 \rfloor} \sum_{ll=0}^{p-2d+2z} \sum_{kk=0}^{z-d+\lfloor p \rfloor 2} \binom{j}{j-z} \frac{p! \delta_{l,ll} \delta_{k,kk}}{d!(p-2d)!} \\ & \times \left(\frac{T_a}{T_{\parallel a}} \right)^{p/2} \left(\frac{T_a}{T_{\perp a}} \right)^z \left(1 - \frac{T_{\parallel a}}{T_a} \right)^d \left(1 - \frac{T_a}{T_{\perp a}} \right)^{j-z} \left(\bar{T}^{-1} \right)_{p-2dz}^{llkk}. \end{aligned} \quad (\text{A } 8)$$

Appendix B. Guiding-centre moments of $\mathcal{A}$

In (2.16), the term $\mathcal{A}$ that ensures phase-space conservation properties for the particle equations of motion is introduced. Here, we present the analytic expressions for its guiding-centre moments $\|\mathcal{A}\|_a^{*pj}$ appearing in (4.12). These are given by

$$\begin{aligned} \|\mathcal{A}\|_a^{*pj} = & \frac{1}{N_a \Omega_a} \sum_{l,k} \left(A_{1a} \mathcal{V}_{lk}^{3pj} + A_{2a} \mathcal{V}_{lk}^{2pj} + A_{3a} \mathcal{V}_{lk}^{1pj} \right. \\ & \left. + A_{4a} \mathcal{V}_{lk}^{1p'j'} \mathcal{M}_{p'j'}^{pj} + A_{5a} \mathcal{M}_{lk}^{pj} + A_{6a} \delta_{pl} \delta_{jk} \right) N_a^{lk}, \end{aligned} \quad (\text{B } 1)$$

with the phase-mixing term

$$\begin{aligned} \mathcal{V}_{lk}^{3pj} = & \left[\sqrt{(p+3)(p+2)(p+1)} \delta_{p+3,l} + 3\sqrt{(p+1)^3} \delta_{p+1,l} \right. \\ & \left. + 3\sqrt{p^3} \delta_{p-1,l} + \sqrt{p(p-1)(p-2)} \delta_{p-3,l} \right] \frac{\delta_{j,k}}{\sqrt{8}}, \end{aligned} \quad (\text{B } 2)$$

and the coefficients A_i

$$A_{1a} = v_{th\parallel a}^3 \mathbf{k}_\perp \cdot \nabla \times \mathbf{b}, \quad (\text{B } 3)$$

$$A_{2a} = v_{th\parallel a}^2 [\mathbf{k}_\perp \cdot (u_{\parallel a} \nabla \times \mathbf{b} + \nabla \times \mathbf{v}_E) + \nabla \times \mathbf{b} \cdot \mathbf{A}], \quad (\text{B } 4)$$

$$A_{3a} = v_{th\parallel a} (u_{\parallel a} \nabla \times \mathbf{b} + \nabla \times \mathbf{v}_E) \cdot \mathbf{A} + v_{th\parallel a}^2 \nabla \times \mathbf{b} \cdot \mathbf{C}, \quad (\text{B } 5)$$

$$A_{4a} = v_{th\parallel a} \frac{T_\perp}{m_a B} \nabla_\perp \mathbf{B} \cdot \nabla \times \mathbf{b}, \quad (\text{B } 6)$$

$$A_{5a} = \frac{T_{\perp a}}{m_a B} \nabla_\perp \mathbf{B} \cdot (u_{\parallel a} \nabla \times \mathbf{b} + \nabla \times \mathbf{v}_E), \quad (\text{B } 7)$$

$$A_{6a} = (v_{th\parallel a} u_{\parallel a} \nabla \times \mathbf{b} + \nabla \times \mathbf{v}_E) \cdot \mathbf{C} \quad (\text{B } 8)$$

with

$$\mathbf{A} = \left[\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{b} \cdot \nabla) \mathbf{v}_E + (\mathbf{v}_E \cdot \nabla) \mathbf{b} + 2u_{\parallel a} v_{th\parallel a} \mathbf{k} \right]_\perp, \quad (\text{B } 9)$$

$$\mathbf{C} = \frac{1}{v_{th\parallel a}} \left[\frac{\partial \mathbf{v}_E}{\partial t} + (\mathbf{v}_E \cdot \nabla) \mathbf{v}_E + u_{\parallel a}^2 \mathbf{k} \right]_\perp. \quad (\text{B } 10)$$

Appendix C. Poisson's equation with collisional effects

To include ϵ_v effects in Poisson's equation, we retain the $l=1$ Bessel term in (5.6), yielding

$$\epsilon_0 \nabla \cdot \mathbf{E} = \sum_a q_a \left[N_a \left(1 + \frac{\mathbf{b} \cdot \nabla \times \mathbf{b}}{\Omega} u_{\parallel a} + \frac{\mathbf{b} \cdot \nabla \times \mathbf{v}_E}{\Omega} \right) + \frac{1}{2m_a} \nabla_\perp^2 \left(\frac{P_{\perp a}}{\Omega_a^2} \right) + 2\pi \int \Gamma_1 [C_{1a} e^{i\alpha} + C_{-1a} e^{-i\alpha}] \frac{B_{\parallel}^*}{m_a} dv_{\parallel} d\mu \right]. \quad (\text{C } 1)$$

The collisional terms $C_{\pm 1a} = \sum_b C_{\pm 1ab}$ (for collisions between species a and b) can be cast in terms of gyrofluid moments N_a^{lk} . For like-species collisions, we use (3.22) to express the collision operator C_{aa0} in (3.21) in terms of fluid moments, together with the property (Ji & Held 2006)

$$\mathbf{P}^l(\mathbf{v}) \cdot \mathbf{T}^{lk} = \mathbf{v}^l \cdot \mathbf{T}^{lk}, \quad (\text{C } 2)$$

which holds for any totally symmetric and traceless tensor $\mathbf{T}^{lk}$. This yields the following form for the lowest-order collision operator (3.22)

$$c_0(f_a^{lkm}, f_a^{nqr}) = f_a M \mathcal{N}_a^{lk} \mathcal{N}_a^{nq} \sum_{u=0}^{\min(2, l, n)} v_{*aa u}^{lm, nr} (c^2) \times \sum_{i=0}^{\min(l, n) - u} d_i^{l-u, n-u} \frac{\mathbf{c}_a^{l+n-2(i+u)}}{c_a^{l+n-2(i+u)}} \cdot \overline{\mathbf{P}^l(\mathbf{b}) \cdot \mathbf{P}^n(\mathbf{b})}, \quad (\text{C } 3)$$

as $\overline{\mathbf{T}}$ means the traceless symmetrization of $\mathbf{T}$. The shifted velocity vector $\mathbf{c}_a = (\mathbf{v} - \mathbf{u}_a)/v_{tha}$, under the transformation of (2.3), with the lowest-order fluid velocity

$\mathbf{u}_a \simeq u_{\parallel a} \mathbf{b} + \mathbf{v}_E$, can be written as

$$\mathbf{c}_a = c_{\parallel a} \mathbf{b} + \frac{c_{\perp a}}{2} (\mathbf{e}^{i\theta} \mathbf{E}_1 + \mathbf{e}^{-i\theta} \mathbf{E}_2), \quad (\text{C4})$$

where $\mathbf{E}_{1,2} = \mathbf{e}_2 \pm i\mathbf{e}_1$. Using the multinomial theorem

$$\left(\sum_{i=1}^m x_i \right)^k = \sum_{a_i \geq 0} \frac{k! \prod_{i=1}^m x_i^{a_i}}{\prod_{i=1}^m a_i!}, \quad (\text{C5})$$

subject to the constraint $\sum_{i=1}^m a_i = k$, we obtain

$$\mathbf{c}_a^k = \sum_{a_1+a_2+a_3=k} \frac{k! (c_{\parallel a})^{a_1}}{a_1! a_2! a_3!} \left(\frac{c_{\perp a}}{2} \right)^{a_2+a_3} \mathbf{e}^{i\theta(a_2-a_3)} \mathbf{b}^{a_1} \mathbf{E}_1^{a_2} \mathbf{E}_2^{a_3}. \quad (\text{C6})$$

We use (C6) to make explicit the dependence of $\mathbf{c}_a$ on θ in (C3). The Fourier components C_{1a} and C_{-1a} correspond to the case $a_2 = a_3 \pm 1$ in the sum in (C6) above, yielding

$$\begin{aligned} C_{\pm 1aa} &= \sum_{l,k,m} \sum_{n,q,r} \frac{L_{km}^l L_{qr}^n}{\sqrt{\sigma_k^l \sigma_q^n}} f_{aM} \mathcal{N}_a^{lk} \mathcal{N}_a^{nq} \sum_{u=0}^{\min(2,l,n)} v_{*aa}^{lm,nr} (c_a^2) \sum_{i=0}^{\min(l,n)-u} d_i^{l-u,n-u} \\ &\times \sum_{\substack{a_1+a_2+a_3=l+n-2(i+u)}} \frac{(l+n-2(i+u))!}{a_1! a_2! a_3!} (c_{\parallel a})^{a_1} \left(\frac{c_{\perp a}}{2} \right)^{a_2+a_3} \delta_{a_2, a_3 \pm 1} \\ &\times \frac{\mathbf{b}^{a_1} \mathbf{E}_1^{a_2} \mathbf{E}_2^{a_3}}{(c_a)^{l+n-2(i+u)}} \cdot \overline{\mathbf{P}^l(\mathbf{b}) \cdot {}^{i+u} \mathbf{P}^n(\mathbf{b})}. \end{aligned} \quad (\text{C7})$$

Assembling the velocity dependent terms of (C7), together with $J_1(k_{\perp} \rho_a) \simeq k_{\perp} v_{tha} c_{\perp a} / (2\Omega_a)$, the velocity integration of the like-species operator in Poisson's equation (C1) is then

$$\begin{aligned} I_{aa}^{\pm} &= \int c_{\perp a} C_{\pm 1aa} \frac{B_{\parallel}^*}{m_a} d\mathbf{v}_{\parallel} d\mu = \sum_{lkm} \sum_{nqr} \sum_{u=0}^{\min(2,l,n)} \sum_{i=0}^{\min(l,n)-u} \sum_{a_1+a_2+a_3=l+n-2(i+u)} \\ &\times \frac{L_{km}^l L_{qr}^n}{\sqrt{\sigma_k^l \sigma_q^n}} \mathcal{N}_a^{lk} \mathcal{N}_a^{nq} d_i^{l-u,n-u} \delta_{a_2, a_3 \pm 1} \frac{(l+n-2(i+u))!}{a_1! a_2! a_3! 2^{a_2+a_3}} \\ &\times \mathbf{b}^{a_1} \mathbf{E}_1^{a_2} \mathbf{E}_2^{a_3} \cdot \overline{\mathbf{P}^l(\mathbf{b}) \cdot {}^{i+u} \mathbf{P}^n(\mathbf{b})} v_{tha}^3 I_{a\pm}, \end{aligned} \quad (\text{C8})$$

where

$$I_{a\pm} = \int f_{Ma} c_{\perp a} (c_{\parallel a})^{a_1} (c_{\perp a})^{2a_2 \pm 1} \frac{v_{*aa}^{lm,nr} (c_a^2) B_{\parallel}^*}{c_a^{l+n-2(i+u)} m_a} d\mathbf{v}_{\parallel} d\mu. \quad (\text{C9})$$

Converting to pitch-angle coordinates $v_{\parallel} = v_{tha} c_{\parallel a} = v_{tha} \xi_a c_a$, $v_{\perp}^2 = v_{tha}^2 c_{\perp a}^2 = c_a^2 (1 - \xi_a^2)$, with the volume element $(B_{\parallel}^* / m_a) d\mu dv_{\parallel} = v_{tha}^3 c_a^2 dc_a d\xi_a$, the integral of (C9) can be

performed analytically, yielding

$$I_{a\pm} = \int_{-1}^1 \xi_a^{a_1} (1 - \xi_a^2)^{a_3 + \sigma_{\pm}} d\xi_a \int f_{Ma} v_{*aau}^{lm,nr} c^{2(a_3 + \sigma_{\pm} + i + u) + a_1 - l - n} \frac{d^3 c}{4\pi} \\ = \frac{((-1)^{a_1} + 1) \Gamma\left(\frac{a_1 + 1}{2}\right) \Gamma(a_3 + \sigma_{\pm} + 1)}{8\pi \Gamma\left(\frac{a_1}{2} + a_3 + \sigma_{\pm} + \frac{3}{2}\right)} C_{*aau}^{a_1 - l - n, a_3 + \sigma_{\pm} + i + u, lm, nr}, \quad (C 10)$$

where $\sigma_{\pm} = (1 \pm 1)/2$ for $C_{\pm laa}$ respectively.

For electron–ion collisions, we take the expression for $C_{ei} = C_{ei}^0 + C_{ei}^1$ given by equations (2.25) and (2.26) respectively. By using the shifted velocity vector $\mathbf{c}_a$ of (C 4), we can proceed as for the like-species operator, yielding the Fourier components $C_{\pm lei} = C_{\pm lei}^0 + C_{\pm lei}^1$, with

$$C_{\pm lei}^0 = - \sum_{l,k} \frac{n_i L_{ei}}{8\pi c_e^3} \frac{l(l+1) f_e \mathcal{N}_e^{lk}}{\sqrt{\sigma_k^l}} \\ \times \sum_{a_1 + a_2 + a_3 = l} \frac{l! (c_{\parallel e})^{a_1}}{a_1! a_2! a_3!} \left(\frac{c_{\perp e}}{2}\right)^{a_2 + a_3} \mathbf{b}^{a_1} \mathbf{E}_1^{a_2} \mathbf{E}_2^{a_3} \cdot \mathbf{P}^l(\mathbf{b}) \delta_{a_2, a_3 \pm 1}, \quad (C 11)$$

and

$$C_{\pm lei}^1 = f_{Me} \frac{n_i L_{ei} m_e}{8\pi c_e^3 T_e} \frac{c_{\perp e}}{2} \mathbf{u}_{ei} \cdot \mathbf{E}_{1,2}. \quad (C 12)$$

The velocity integration of $C_{\pm lei}^0$ and $C_{\pm lei}^1$ are given by

$$I_{ei}^{0\pm} = \int c_{\perp e} C_{\pm lei}^0 \frac{B_{\parallel}^*}{m_a} dv_{\parallel} d\mu \\ = - \sum_{l,k} \sum_{a_1 + a_2 + a_3 = l} \delta_{a_2, a_3 \pm 1} \frac{v_{ei} n_e v_{the}^3 \mathcal{N}_e^{lk}}{16\pi^{3/2} \sqrt{\sigma_k^l} 2^{a_2 + a_3}} \frac{l(l+1)l!}{a_1! a_2! a_3!} \\ \times \frac{\Gamma\left(\frac{a_1 + a_3 + \sigma_{\pm}}{2}\right) \Gamma\left(\frac{a_1 + 1}{2}\right) \Gamma(a_3 + \sigma_{\pm} + 1)}{\Gamma\left(\frac{3}{2} + \frac{a_1}{2} + a_3 + \sigma_{\pm}\right)}, \quad (C 13)$$

and

$$I_{ei}^{1\pm} = \int c_{\perp e} C_{\pm lei}^1 \frac{B_{\parallel}^*}{m_a} dv_{\parallel} d\mu = \frac{n_e v_{ei}}{6\sqrt{\pi}} v_{the}^2 \mathbf{u}_{ei} \cdot \mathbf{E}_{1,2}, \quad (C 14)$$

respectively. Ion–electron collisions are neglected due to the smallness of the electron to ion mass ratio. Poisson’s equation including ϵ^2 and $\epsilon_v \epsilon$ effects then reads

$$\epsilon_0 \nabla \cdot \mathbf{E} = \sum_a q_a \left[N_a \left(1 + \frac{\mathbf{b} \cdot \nabla \times \mathbf{b}}{\Omega_a} u_{\parallel a} + \frac{\mathbf{b} \cdot \nabla \times \mathbf{v}_E}{\Omega_a} \right) \right. \\ \left. + \frac{1}{2m_a} \nabla_{\perp}^2 \left(\frac{P_{\perp a}}{\Omega_a^2} \right) + \sum_b \frac{\pi v_{tha}}{\Omega_a} \int k_{\perp} (e^{i\alpha} I_{ab}^+ + e^{-i\alpha} I_{ab}^-) e^{i\mathbf{k} \cdot \mathbf{x}} d^3 \mathbf{k} \right]. \quad (C 15)$$

Appendix D. Expressions for the moments of the collision operator

In the present appendix, we present the expressions for the guiding-centre moments of the collision operator relevant for the fluid model in § 6. The collision operator moments satisfy particle conservation

$$C_{ab}^{00} = 0, \quad (\text{D } 1)$$

and momentum conservation at lowest order

$$C_{aa}^{10} = 0, \quad (\text{D } 2)$$

$$C_{ei}^{10} = -\frac{m_i}{m_e} \frac{v_{th\parallel i}}{v_{th\parallel e}} C_{ie}^{10} + O(m_e/m_i). \quad (\text{D } 3)$$

Both the like species and electron–ion collisions satisfy energy conservation exactly, while the ion–electron operator satisfies (D 4) at zeroth order in δ_a

$$T_{\parallel a} C_{ab}^{20} - \sqrt{2} T_{\perp a} C_{ab}^{01} = 0. \quad (\text{D } 4)$$

The remaining moments C_{ab}^{pj} in the linear transport regime with $\Delta T_a/T_a = (T_{\parallel a} - T_{\perp a})/T_a \sim N^{11} \sim N^{30} \sim (u_{\parallel e} - u_{\parallel i})/v_{the} \sim \delta_a$, for ion–electron collisions are given by

$$C_{ie}^{10} = -\frac{m_e}{m_i} \frac{v_{th\parallel i}}{v_{th\parallel e}} C_{ei}^{10}, \quad (\text{D } 5)$$

$$C_{ie}^{20} = \sqrt{2} v_{ei} \frac{m_e}{m_i} \left(\frac{T_e - T_i}{T_i} \right) - \frac{2\sqrt{2} v_{ei} m_e}{3} \frac{T_e}{m_i} \frac{\Delta T_i}{T_i}, \quad (\text{D } 6)$$

$$C_{ie}^{01} = -2 v_{ei} \frac{m_e}{m_i} \left(\frac{T_e - T_i}{T_i} \right) - \frac{2 v_{ei} m_e}{3} \frac{T_e}{m_i} \frac{\Delta T_i}{T_i}, \quad (\text{D } 7)$$

$$C_{ie}^{30} = -v_{ei} \sqrt{\frac{3}{2}} \frac{m_e}{m_i} \frac{Q_{\parallel i}}{n T_i v_{thi}}, \quad (\text{D } 8)$$

$$C_{ie}^{11} = 3 v_{ei} \frac{m_e}{m_i} \frac{Q_{\perp i}}{n T_i v_{thi}}, \quad (\text{D } 9)$$

for electron–ion collisions

$$C_{ei}^{10} = -\frac{\sqrt{2} v_{ei}}{6\pi^{3/2}} \frac{u_{\parallel e} - u_{\parallel i}}{v_{the}} + \frac{\sqrt{2} v_{ei}}{10\pi^{3/2}} \frac{Q_{\parallel e} + 2Q_{\perp e}}{n T_e v_{the}}, \quad (\text{D } 10)$$

$$C_{ei}^{20} = -\frac{2\sqrt{2} v_{ei}}{15\pi^{3/2}} \frac{\Delta T_e}{T_e}, \quad (\text{D } 11)$$

$$C_{ei}^{30} = \frac{\sqrt{3} v_{ei}}{10\pi^{3/2}} \frac{u_{\parallel e} - u_{\parallel i}}{v_{the}} - \frac{v_{ei}}{70\sqrt{3}\pi^{3/2}} \frac{31Q_{\parallel e} - 2Q_{\perp e}}{n T_e v_{the}}, \quad (\text{D } 12)$$

$$C_{ei}^{11} = \frac{v_{ei}}{5\sqrt{2}\pi^{3/2}} \frac{u_{\parallel e} - u_{\parallel i}}{v_{the}} + \frac{v_{ei}}{150\sqrt{2}\pi^{3/2}} \frac{Q_{\parallel e} - 94Q_{\perp e}}{n T_e v_{the}}, \quad (\text{D } 13)$$

and for like-species collisions

$$C_{aa}^{20} = 0, \quad (\text{D } 14)$$

$$C_{aa}^{30} = -\frac{2\sqrt{2}}{125\sqrt{3}\pi^{3/2}} \frac{v_{aa}}{nT_a v_{tha}} (19Q_{\parallel a} - 7Q_{\perp a}), \quad (\text{D } 15)$$

$$C_{aa}^{11} = -\frac{2}{375\pi^{3/2}} \frac{v_{aa}}{nT_a v_{tha}} (7Q_{\parallel a} - 121Q_{\perp a}). \quad (\text{D } 16)$$

REFERENCES

- ABEL, I. G., PLUNK, G. G., WANG, E., BARNES, M., COWLEY, S. C., DORLAND, W. & SCHEKOCHIHIN, A. A. 2013 Multiscale gyrokinetics for rotating tokamak plasmas: fluctuations, transport and energy flows. *Rep. Prog. Phys.* **76** (11), 116201.
- ABRAMOWICZ, M., CZERNY, B., LASOTA, J. & SZUSZKIEWICZ, E. 1988 Slim accretion disks. *Astrophys. J.* **332**, 646.
- ABRAMOWITZ, M., STEGUN, I. & MILLER, D. 1965 *Handbook of Mathematical Functions With Formulas, Graphs and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series No. 55, vol. 32. United States Department of Commerce, National Institute of Standards and Technology (NBS).
- AGOSTINI, M., TERRY, J., SCARIN, P. & ZWEBEN, S. 2011 Edge turbulence in different density regimes in Alcator C-Mod experiment. *Nucl. Fusion* **51** (5), 053020.
- ARMSTRONG, T. 1967 Numerical Studies of the Nonlinear Vlasov Equation. *Phys. Fluids* **10** (6), 1269.
- BALESCU, R. 1988 *Transport Processes in Plasmas*. North-Holland.
- BATISHCHEV, O. V., KRASHENINNIKOV, S. I., CATTO, P. J., BATISHCHEVA, A. A., SIGMAR, D. J., XU, X. Q., BYERS, J. A., ROGNLIEN, T. D., COHEN, R. H., SHOUCRI, M. M. *et al.* 1997 Kinetic effects in tokamak scrape-off layer plasmas. *Phys. Plasmas* **4** (5/2), 1672–1680.
- BATTAGLIA, D. J., BURRELL, K. H., CHANG, C. S., KU, S., DEGRASSIE, J. S. & GRIERSON, B. A. 2014 Kinetic neoclassical transport in the H-mode pedestal. *Phys. Plasmas* **21** (7), 072508.
- BEER, M. & HAMMETT, G. 1996 Toroidal gyrofluid equations for simulations of tokamak turbulence. *Phys. Plasmas* **3** (1996), 4046.
- BRAGINSKII, S. I. 1965 *Transport Processes in a Plasma*. Consultants Bureau.
- BRIZARD, A. 1992 Nonlinear gyrofluid description of turbulent magnetized plasmas. *Phys. Fluids B* **4** (1992), 1213.
- CAMPOREALE, E., DELZANNO, G. L., BERGEN, B. K. & MOULTON, J. D. 2016 On the velocity space discretization for the Vlasov–Poisson system: comparison between implicit Hermite spectral and Particle-in-Cell methods. *Comput. Phys. Commun.* **198**, 47–58.
- CARRALERO, D., BIRKENMEIER, G., MÜLLER, H., MANZ, P., DEMARNE, P., MÜLLER, S., REIMOLD, F., STROTH, U., WISCHMEIER, M. & WOLFRUM, E. 2014 An experimental investigation of the high density transition of the scrape-off layer transport in ASDEX Upgrade. *Nucl. Fusion* **54** (12), 123005.
- CARRERAS, B. 2005 Plasma edge cross-field transport: experiment and theory. *J. Nucl. Mater.* **337–339** (1–3), 315–321.
- CARY, J. & BRIZARD, A. 2009 Hamiltonian theory of guiding-center motion. *Rev. Mod. Phys.* **81** (2), 693–738.
- CATTO, P. J. & SIMAKOV, A. N. 2004 A drift ordered short mean free path description for magnetized plasma allowing strong spatial anisotropy. *Phys. Plasmas* **11** (1), 90–102.
- CHANG, C., KU, S., TYNAN, G., HAGER, R., CHURCHILL, R., CZIEGLER, I., GREENWALD, M., HUBBARD, A. & HUGHES, J. 2017 Fast low-to-high confinement mode bifurcation dynamics in a tokamak edge plasma gyrokinetic simulation. *Phys. Rev. Lett.* **118** (17), 175001.
- CHAPMAN, S. 1962 *The Mathematical Theory of Non-Uniform Gases*.
- D’IPPOLITO, D., MYRA, J. & KRASHENINNIKOV, S. 2002 Cross-field blob transport in tokamak scrape-off-layer plasmas. *Phys. Plasmas* **9** (1), 222.
- D’IPPOLITO, D., MYRA, J. & ZWEBEN, S. 2011 Convective transport by intermittent blob-filaments: comparison of theory and experiment. *Phys. Plasmas* **18** (6), 060501.

- DORLAND, W. & HAMMETT, G. W. 1993 Gyrofluid turbulence models with kinetic effects. *Phys. Fluids B-Plasma Phys.* **5** (1993), 812–835.
- DUBIN, D., KROMMES, J. & OBERMAN, C. 1983 Nonlinear gyrokinetic equations. *Phys. Fluids* **26** (12), 3524.
- DUDSON, B., UMANSKY, M., XU, X., SNYDER, P. & WILSON, H. 2009 BOUT++: a framework for parallel plasma fluid simulations. *Comput. Phys. Commun.* **180** (9), 1467–1480.
- EASY, L., MILITELLO, F., OMOTANI, J., DUDSON, B., HAVLÍČKOVÁ, E., TAMAIN, P., NAULIN, V. & NIELSEN, A. H. 2014 Three dimensional simulations of plasma filaments in the scrape off layer: a comparison with models of reduced dimensionality. *Phys. Plasmas* **21** (12), 122515.
- ENDLER, M., NIEDERMEYER, H., GIANNONE, L., KOLZHAUER, E., RUDYJ, A., THEIMER, G. & TSOIS, N. 1995 Measurements and modelling of electrostatic fluctuations in the scrape-off layer of ASDEX. *Nucl. Fusion* **35** (11), 1307–1339.
- ERENTS, S., CHANKIN, A., MATTHEWS, G. & STANGEBY, P. 2000 Parallel flow in the JET scrape-off layer. *Plasma Phys. Control. Fusion* **42** (8), 905–915.
- GARCIA, O., HORACEK, J. & PITTS, R. A. 2015 Intermittent fluctuations in the TCV scrape-off layer. *Nucl. Fusion* **55** (6), 062002.
- GRAD, H. 1949 On the kinetic theory of rarefied gases. *Commun. Pure Appl. Math.* **2** (4), 331–407.
- GRANT, F. C. & FEIX, M. C. 1967 Fourier–Hermite solutions of the Vlasov equations in the linearized limit. *Phys. Fluids* **10** (4), 696–702.
- GROŠELJ, D., CERRI, S. S., NAVARRO, A. B., WILLMOTT, C., TOLD, D., LOUREIRO, N. F., CALIFANO, F. & JENKO, F. 2017 Fully kinetic versus reduced-kinetic modeling of collisionless plasma turbulence. *Astrophys. J.* **847** (1), 28.
- GRULKE, O., TERRY, J. L., CZIEGLER, I., LABOMBARD, B. & GARCIA, O. E. 2014 Experimental investigation of the parallel structure of fluctuations in the scrape-off layer of Alcator C-Mod. *Nucl. Fusion* **54** (4), 43012.
- HAHM, T. 1988 Nonlinear gyrokinetic equations for tokamak microturbulence. *Phys. Fluids* **31** (9), 2670–2673.
- HAHM, T. S., WANG, L. & MADSEN, J. 2009 Fully electromagnetic nonlinear gyrokinetic equations for tokamak edge turbulence. *Phys. Plasmas* **16** (2), 022305.
- HALPERN, F., RICCI, P., JOLLIET, S., LOIZU, J., MORALES, J., MOSETTO, A., MUSIL, F., RIVA, F., TRAN, T. & WERSAL, C. 2016 The GBS code for tokamak scrape-off layer simulations. *J. Comput. Phys.* **315**, 388–408.
- HAMMETT, G., DORLAND, W. & PERKINS, F. 1992 Fluid models of phase mixing, Landau damping, and nonlinear gyrokinetic dynamics. *Phys. Fluids B* **4** (7), 2052–2061.
- HAMMETT, G. W., BEER, M. A., DORLAND, W. D., COWLEY, S. C. & SMITH, S. A. 1993 Developments in the gyrofluid approach to Tokamak turbulence simulations. *Plasma Phys. Control. Fusion* **35** (8), 973–985.
- HATCH, D. R., JENKO, F., NAVARRO, A. B., BRATANOV, V., TERRY, P. W. & PUESCHEL, M. J. 2016 Linear signatures in nonlinear gyrokinetics: interpreting turbulence with pseudospectra. *New J. Phys.* **18** (7), 075018.
- HAZELTINE, R. D. 1998 Transport theory in the collisionless limit. *Phys. Plasmas* **5** (9), 3282.
- HAZELTINE, R. D. & MEISS, J. D. 2003 *Plasma Confinement*. Dover.
- HELANDER, P. & SIGMAR, D. 2005 *Collisional Transport in Magnetized Plasmas*, Cambridge Monographs on Plasma Physics. Cambridge University Press.
- HELD, M., WIESENBERGER, M., MADSEN, J. & KENDL, A. 2016 The influence of temperature dynamics and dynamic finite ion Larmor radius effects on seeded high amplitude plasma blobs. *Nucl. Fusion* **56** (12), 126005.
- HIDALGO, C., GONÇALVES, B., PEDROSA, M., CASTELLANO, J., ERENTS, K., FRAGUAS, L., HRON, M., JIMENEZ, J., MATTHEWS, G., VAN MILLIGEN, B. *et al.* 2002 Empirical similarity in the probability density function of turbulent transport in the edge plasma region in fusion plasmas. *Plasma Phys. Control. Fusion* **44**, 1557–1564.
- HINTON, F. & HAZELTINE, R. D. 1976 Theory of plasma transport in toroidal confinements systems. *Rev. Mod. Phys.* **48** (2), 239–308.

- HIRVIJOKI, E., LINGAM, M., PFEFFERLÉ, D., COMISSO, L., CANDY, J. & BHATTACHARJEE, A. 2016 Fluid moments of the nonlinear Landau collision operator. *Phys. Plasmas* **23** (8), 1–10.
- JACKSON, J. 1998 Classical electrodynamics, 3rd Edition. *Am. J. Phys.* **841** (1999), 159.
- Ji, J.-Y. & HELD, E. D. 2006 Exact linearized Coulomb collision operator in the moment expansion. *Phys. Plasmas* **13** (10), 102103.
- Ji, J.-Y. & HELD, E. D. 2008 Landau collision operators and general moment equations for an electron-ion plasma. *Phys. Plasmas* **15** (10), 102101.
- Ji, J.-Y. & HELD, E. D. 2009 Full Coulomb collision operator in the moment expansion. *Phys. Plasmas* **16** (10), 102108.
- Ji, J.-Y. & HELD, E. D. 2010 Analytical solution of the kinetic equation for a uniform plasma in a magnetic field. *Phys. Rev. E* **82** (1), 1–5.
- JORGE, R., RICCI, P., HALPERN, F., LOUREIRO, N. F. & SILVA, C. 2016 Plasma turbulence in the scrape-off layer of the ISTTOK tokamak. *Phys. Plasmas* **23** (10), 102511.
- KOČAN, M., GUNN, J., CARPENTIER-CHOUGHANA, S., HERRMANN, A., KIRK, A., KOMM, M., MÜLLER, H., PASCAL, J., PITTS, R., ROHDE, V. *et al.* 2011 Measurements of ion energies in the tokamak plasma boundary. *J. Nucl. Mater.* **415** (1 suppl), S1133–S1138.
- KROMMES, J. A. 2013 The physics of the second-order gyrokinetic magnetohydrodynamic Hamiltonian: μ conservation, Galilean invariance, and ponderomotive potential. *Phys. Plasmas* **20** (12), 124501.
- LABOMBARD, B., BOIVIN, R., GREENWALD, M., HUGHES, J., LIPSCHULTZ, B., MOSSESIAN, D., PITCHER, C., TERRY, J. & ZWEBEN, S. 2001 Particle transport in the scrape-off layer and its relationship to discharge density limit in Alcator C-Mod. *Phys. Plasmas* **8** (5 II), 2107–2117.
- LABOMBARD, B., HUGHES, J., MOSSESIAN, D., GREENWALD, M., LIPSCHULTZ, B., TERRY, J. & THE ALCATOR C-MOD TEAM 2005 Evidence for electromagnetic fluid drift turbulence controlling the edge plasma state in the Alcator C-Mod tokamak. *Nucl. Fusion* **45** (12), 1658.
- LEE, W. 1983 Gyrokinetic approach in particle simulation. *Phys. Fluids* **26** (2), 556.
- LEONARD, A. W. 2014 Edge-localized-modes in tokamaks. *Phys. Plasmas* **21** (9), 090501.
- LIANG, Y., KOSLOWSKI, H. R., THOMAS, P. R., NARDON, E., ALPER, B., ANDREW, P., ANDREW, Y., ARNOUX, G., BARANOV, Y., BÉCOULET, M. *et al.* 2007 Active control of type-I edge-localized modes with $n=1$ perturbation fields in the JET tokamak. *Phys. Rev. Lett.* **98** (26), 265004.
- LITTLEJOHN, R. G. 1983 Variational principles of guiding centre motion. *J. Plasma Phys.* **29** (1), 111–125.
- LÖNNROTH, J. S., BATEMAN, G., BÉCOULET, M., BEYER, P., CORRIGAN, G., FIGARELLA, C., FUNDAMENSKI, W., GARCIA, O. E., GARBET, X., HUYSMANS, G. *et al.* 2006 Integrated ELM modelling. *Contrib. Plasma Phys.* **46** (7–9), 726–738.
- LOUREIRO, N. F., DORLAND, W., FAZENDEIRO, L., KANEKAR, A., MALLET, A., VILELAS, M. S. & ZOCCO, A. 2016 Viriato: A Fourier–Hermite spectral code for strongly magnetized fluid-kinetic plasma dynamics. *Comput. Phys. Commun.* **206**, 45–63.
- LOUREIRO, N. F., SCHEKOCIHIN, A. A. & ZOCCO, A. 2013 Fast collisionless reconnection and electron heating in strongly magnetized plasmas. *Phys. Rev. Lett.* **111** (2), 025002.
- MADSEN, J. 2013 Full-F gyrofluid model. *Phys. Plasmas* **20** (7), 072301.
- MADSEN, J., NAULIN, V., NIELSEN, A. H. & RASMUSSEN, J. J. 2016 Collisional transport across the magnetic field in drift-fluid models. *Phys. Plasmas* **23** (3), 032306.
- MANDELL, N. R., DORLAND, W. & LANDREMAN, M. 2017 Laguerre–Hermite pseudo-spectral velocity formulation of gyrokinetics. [arXiv:1708.04029](https://arxiv.org/abs/1708.04029).
- MARTIN, Y., TAKIZUKA, T. & GROUP, THE ITPA CDBM H-MODE THRESHOLD DATA 2008 Power requirement for accessing the H-mode in ITER. *J. Phys.: Conf. Ser.* **123**, 012033.
- MIKHAILOVSKII, A. & TSYPIN, V. 1971 Transport equations and gradient instabilities in a high pressure collisional plasma. *Plasma Phys.* **13** (9), 785–798.
- MOSETTO, A., HALPERN, F., JOLLIET, S., LOIZU, J. & RICCI, P. 2015 Finite ion temperature effects on scrape-off layer turbulence. *Phys. Plasmas* **22** (1), 0–17.
- MYRA, J., DAVIS, W., D’IPPOLITO, D., LABOMBARD, B., RUSSELL, D., TERRY, J. & ZWEBEN, S. 2013 Edge sheared flows and the dynamics of blob-filaments. *Nucl. Fusion* **53** (7), 073013.

- NESPOLI, F., FURNO, I., LABIT, B., RICCI, P., AVINO, F., HALPERN, F., MUSIL, F. & RIVA, F. 2017 Blob properties in full-turbulence simulations of the TCV scrape-off layer. *Plasma Phys. Control. Fusion* **59** (5), 055009.
- NG, C. S., BHATTACHARJEE, A. & SKIFF, F. 1999 Kinetic eigenmodes and discrete spectrum of plasma oscillations in a weakly collisional plasma. *Phys. Rev. Lett.* **83** (10), 1974–1977.
- OMOTANI, J., DUDSON, B., HAVLÍČKOVÁ, E. & UMANSKY, M. 2015 Non-local parallel transport in BOUT++. *J. Nucl. Mater.* **463**, 769–772.
- PARKER, J. T. 2016 Gyrokinetic simulations of fusion plasmas using a spectral velocity space representation. [arXiv:1603.04727](https://arxiv.org/abs/1603.04727).
- PARKER, J. T. & DELLAR, P. J. 2015 Fourier–Hermite spectral representation for the Vlasov–Poisson system in the weakly collisional limit. *J. Plasma Phys.* **2** (2011), 1–36.
- PAŠKAUSKAS, R. & DE NINNO, G. 2009 Lyapunov stability of Vlasov equilibria using Fourier–Hermite modes. *Phys. Rev. E* **80** (3), 036402.
- PITTS, R., ALBERTI, S., BLANCHARD, P., HORACEK, J., REIMERDES, H. & STANGEBY, P. 2003 ELM driven divertor target currents on TCV. *Nucl. Fusion* **43** (10), 1145–1166.
- QIN, H., COHEN, R., NEVINS, W. & XU, X. 2007 Geometric gyrokinetic theory for edge plasmas. *Phys. Plasmas* **14** (5), 056110.
- RIBEIRO, T. & SCOTT, B. 2008 Gyrofluid turbulence studies of the effect of the poloidal position of an axisymmetric Debye sheath. *Plasma Phys. Control. Fusion* **50** (5), 055007.
- RITZ, C., BROWER, D., RHODES, T., BENGTSON, R., LEVINSON, S., LUHMANN, N., PEEBLES, W. & POWERS, E. 1987 Characterization of tokamak edge turbulence by far-infrared laser scattering and Langmuir probes. *Nucl. Fusion* **27** (7), 1125–1134.
- ROSSEL, J., MORET, J., CODA, S., SAUTER, O., GOODMAN, T., FELICI, F., TESTA, D. & MARTIN, Y. 2012 Edge-localized mode control by electron cyclotron waves in a tokamak plasma. *Nucl. Fusion* **52** (3), 2–6.
- SCHEKOCHIHIN, A. A., PARKER, J. T., HIGHCOCK, E. G., DELLAR, P. J., DORLAND, W. & HAMMETT, G. 2016 Phase mixing versus nonlinear advection in drift-kinetic plasma turbulence. *J. Plasma Phys.* **82** (02), 905820212.
- SERIANNI, G., AGOSTINI, M., ANTONI, V., CAVAZZANA, R., MARTINES, E., SATTIN, F., SCARIN, P., SPADA, E., SPOLAORE, M., VIANELLO, N. *et al.* 2007 Coherent structures and transport properties in magnetized plasmas. *Plasma Phys. Control. Fusion* **49** (12B), B267–B280.
- SHI, E., HAKIM, A. & HAMMETT, G. 2015 A gyrokinetic one-dimensional scrape-off layer model of an edge-localized mode heat pulse. *Phys. Plasmas* **22** (2), 022504.
- SHI, E. L., HAMMETT, G. W., STOLTZFUS-DUECK, T. & HAKIM, A. 2017 Gyrokinetic continuum simulation of turbulence in a straight open-field-line plasma. *J. Plasma Phys.* **83** (03), 905830304.
- SILVA, C., FIGUEIREDO, H., DUARTE, P. & FERNANDES, H. 2011 Poloidal asymmetries in the ISTTOK edge plasma. *J. Nucl. Mater.* **415** (1), S455–S458.
- SNYDER, P. & HAMMETT, G. 2001 A Landau fluid model for electromagnetic plasma microturbulence. *Phys. Plasmas* **8** (7), 3199–3216.
- STANGEBY, P. 2000 *The Plasma Boundary of Magnetic Fusion Devices*, Series in Plasma Physics, vol. 43. CRC Press.
- SUGAMA, H., WATANABE, T.-H. & HORTON, W. 2001 Collisionless kinetic-fluid closure and its application to the three-mode ion temperature gradient driven system. *Phys. Plasmas* **8** (6), 2617–2628.
- TAMAIN, P., GHENDRIH, P., TSITRONE, E., SARAZIN, Y., GARBET, X., GRANDGIRARD, V., GUNN, J., SERRE, E., CIRAULO, G. & CHIAVASSA, G. 2009 3D modelling of edge parallel flow asymmetries. *J. Nucl. Mater.* **390–391** (1), 347–350.
- TASSI, E. 2016 Hamiltonian reduced fluid model for plasmas with temperature and heat flux anisotropies. *Theor. Math. Phys.* **188** (3), 1377–1393.
- TERRY, J. L., ZWEBEN, S. J., UMANSKY, M. V., CZIEGLER, I., GRULKE, O., LABOMBARD, B. & STOTLER, D. P. 2009 Spatial structure of scrape-off-layer filaments near the midplane and X-point regions of Alcator-C-Mod. *J. Nucl. Mater.* **390–391** (1), 339–342.

- TSKHAKAYA, D. 2012 On recent massively parallelized PIC simulations of the SOL. *Contrib. Plasma Phys.* **52** (5–6), 490–499.
- TSKHAKAYA, D., SUBBA, F., BONNIN, X., COSTER, D. P., FUNDAMENSKI, W. & PITTS, R. A. 2008 On kinetic effects during parallel transport in the SOL. *Contrib. Plasma Phys.* **48** (1–3), 89–93.
- WOOTTON, A., CARRERAS, B., MATSUMOTO, H., MCGUIRE, K., PEEBLES, W., RITZ, C., TERRY, P. & ZWEBEN, S. 1990 Fluctuations and anomalous transport in tokamaks. *Phys. Fluids B* **2** (1990), 2879.
- XU, G., NAULIN, V., FUNDAMENSKI, W., HIDALGO, C., ALONSO, J., SILVA, C., GONÇALVES, B., NIELSEN, A., JUUL RASMUSSEN, J., KRASHENINNIKOV, S. *et al.* 2009 Blob/hole formation and zonal-flow generation in the edge plasma of the JET tokamak. *Nucl. Fusion* **49** (9), 092002.
- XU, X., XIONG, Z., DORR, M., HITTINGER, J., BODI, K., CANDY, J., COHEN, B., COHEN, R., COLELLA, P., KERBEL, G. *et al.* 2007 Edge gyrokinetic theory and continuum simulations. *Nucl. Fusion* **47**, 809–816.
- ZEILER, A., DRAKE, J. F. & ROGERS, B. 1997 Nonlinear reduced Braginskii equations with ion thermal dynamics in toroidal plasma. *Phys. Plasmas* **4** (1997), 2134.
- ZOCCO, A., LOUREIRO, N. F., DICKINSON, D., NUMATA, R. & ROACH, C. M. 2015 Kinetic microtearing modes and reconnecting modes in strongly magnetised slab plasmas. *Plasma Phys. Control. Fusion* **57** (6), 065008.
- ZOCCO, A. & SCHEKOCIHIN, A. A. 2011 Reduced fluid-kinetic equations for low-frequency dynamics, magnetic reconnection, and electron heating in low-beta plasmas. *Phys. Plasmas* **18** (10), 102309.
- ZWEBEN, S., BOEDO, J., GRULKE, O., HIDALGO, C., LABOMBARD, B., MAQUEDA, R., SCARIN, P. & TERRY, J. 2007 Edge turbulence measurements in toroidal fusion devices. *Plasma Phys. Control. Fusion* **49** (7), S1–S23.
- ZWEBEN, S., MAQUEDA, R., STOTLER, D., KEESEE, A., BOEDO, J., BUSH, C., KAYE, S., LEBLANC, B., LOWRANCE, J., MASTROCOLA, V. *et al.* 2004 High-speed imaging of edge turbulence in NSTX. *Nucl. Fusion* **44** (1), 134–153.